Supersymmetry &

Superspace

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Outline

- Notation & Conventions $D=4,5$
- Algebra & Representations $N=0,1,2$
- Fields & Transformations
- Cosets & Covariant Derivatives
  - General
  - $N=0$ $N=1$ $N=2$
- Component Lagrangians

- $N=1$ superspace
- $N=2$ in terms of $N=1$ superspace
- $N=2$ harmonic & projective superspace

Very formal. Applications to others ....

Notation & Conventions

$D = 4, 5$

$x^m \quad x^5 \qquad m = 0,1,2,3 \quad 5$

$\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$

- $D=4$ spinors $\qquad SL(2,\mathbb{C})$ double cover of Lorentz.

$\psi_\alpha \to M_\alpha{}^\beta \psi_\beta \qquad \bar{\psi}_{\dot\alpha} \to m_{\dot\alpha}{}^{\dot\beta} \psi_\beta$

$\alpha = 1,2. \quad \psi$ complex. $\quad M \in SL(2,\mathbb{C})$

Two-component spinor notation.

$\epsilon^{\alpha\beta} \quad \epsilon_{\alpha\beta} \quad \epsilon^{\dot\alpha\dot\beta} \quad \epsilon_{\dot\alpha\dot\beta}$ invariant tensors.

$\psi_\alpha \psi_\beta \epsilon^{\alpha\beta} \neq 0,$ invariant!

- Let $\sigma^a_{\alpha\dot\alpha}$ be basis for 2×2 Hermitian matrices σ^0 σ^i

$$P_{\alpha\dot\alpha} = P_a \sigma^a_{\alpha\dot\alpha}$$ Hermitian.
(useful for twistors)

$$P \rightarrow M P M^{+} \quad \text{under} \quad SL(2,\mathbb{C})$$

$$\det P = P_0^2 - P_i^2 \quad \text{invariant!} \quad \det M = 1.$$

- Define $\bar\sigma^{a\dot\alpha\alpha} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^a_{\beta\dot\beta}$

$$\left(\sigma^a\bar\sigma^b + \sigma^b\bar\sigma^a\right)_\alpha{}^\beta = -2\eta^{ab}\delta_\alpha{}^\beta$$

$$\sigma^{ab}{}_\alpha{}^\beta = \frac{1}{4}\left(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a\right)_\alpha{}^\beta$$

<u>Check</u>: $\epsilon^{abcd}\sigma_{cd} = -2i\,\sigma^{ab}$ $(*)$

<u>Exercise</u>: Use $(*)$ to write the usual Lorentz parameter ω_{ab} in terms of $M_\alpha{}^\beta$!

- Dirac spinors

$$\Psi_{D\hat{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \qquad \bar{\Psi}_D^{\hat{\alpha}} = \left(\chi^\alpha, \bar{\psi}_{\dot{\alpha}}\right)$$

$$\gamma^{a}{}_{\hat{\alpha}}{}^{\hat{\beta}} = \left(\begin{array}{c|c} 0 & \sigma^a \\ \hline \bar{\sigma}^a & 0 \end{array}\right)_{\hat{\alpha}}{}^{\hat{\beta}} \qquad \hat{\alpha} = (\alpha\uparrow, \dot{\alpha}\downarrow)$$

$$\Psi_{M\hat{\alpha}} = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \qquad \Psi_{W\hat{\alpha}} = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}$$

Never worry about Weyl, Majorana, Dirac, pseudo-Dirac ... again!

And in D=5

$$\gamma^{5}{}_{\hat{\alpha}}{}^{\hat{\beta}} = \left(\begin{array}{c|c} -i & \\ \hline & i \end{array}\right)_{\hat{\alpha}}{}^{\hat{\beta}}$$

Algebra & Representations

$$[P_a, P_b] = 0 \qquad D=4.$$

$$[M_{ab}, P_c] = -i(\eta_{bc} P_a - \eta_{ac} P_b)$$

$$[M_{ab}, M_{cd}] = -i(\eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc})$$

$$\{Q_\alpha^i, \bar{Q}_{\dot\beta j}\} = 2\delta^i_j \sigma^a_{\alpha\dot\beta} P_a \qquad i=1\ldots N$$

$$\{Q_\alpha^i, Q_\beta^j\} = \epsilon_{\alpha\beta} Z^{[ij]}$$

$$[M_{ab}, Q_\alpha^i] = \frac{i}{2} \sigma_{ab\alpha}{}^\beta Q_\beta^i$$

Q's transform in fundamental of U(N)

Z's commute with everything!

U(N) is "R symmetry group"

Z's are "central charges".

Most general spacetime algebra consistent w/ relativistic QFT!

We want to represent algebra on states in a relativistic QFT.

Let's find the reps.

Theorem: There are no finite-dimensional unitary representations of noncompact groups.

Wigner: OK, but who cares? Wigner method of induced representations is very physical.

- Pick preferred frame
- Classify states,
- Then rotate, translate & boost!

- Massive particles. Pick rest frame. Classify particles by their mass and spin. Unitary reps of compact "little group" $SO(D-1)$.

$\boxed{\text{DOF}}$ Massive

	scalar	spinor	vector
$D=4$ $SO(3)$	1	2	3
$D=5$ $SO(4)=SU(2)\times SU(2)$	1	4 $(0,2)\oplus(2,0)$	4 $(2,2)$

- Massless particles. Pick lightlike frame. Classify particles by their helicity. Little group is $SO(D-2)$

$\boxed{\text{DOF}}$ Massless

	scalar	spinor	vector
$D=4$ $SO(2)$	1	2	2
$D=5$ $SO(3)$	1	4	3

Now SUSY

- Massive — Use rest frame

$$\{ Q^i_\alpha , \bar{Q}_{\dot\beta j} \} = 2m\, \delta_{\alpha\dot\beta}\, \delta^i_j$$

$\uparrow$ spin

$|\Omega_j\rangle$ — state of spin j

$a^{+i}_\alpha |\Omega_j\rangle$ — N states of spin $j \pm \frac{1}{2}$

etc.

$\boxed{N=1}$

$$|\Omega_j\rangle$$

$$a^+_\alpha |\Omega_j\rangle$$

$$a^+_\alpha a^+_\beta |\Omega_j\rangle = \epsilon_{\alpha\beta}\, (a^+ a^+)\, |\Omega_j\rangle$$

spins $\quad j \quad j\pm\frac{1}{2} \quad j$

	Ω_0	$\Omega_{\frac{1}{2}}$
0	2	1
$\frac{1}{2}$	1	2
1		1

+ higher spins...

B-F symmetry!

$\boxed{N=2}$

$$|\Omega_0\rangle$$

$$a^{+i}_{\alpha}|\Omega_0\rangle$$

$$a^{+[i}_{(\alpha}a^{+j]}_{\beta)}|\Omega_0\rangle \qquad a^{+(i}_{[\alpha}a^{+j)}_{\beta]}|\Omega_0\rangle$$

$$a^+\,a^+\,a^+|\Omega\rangle$$

$$a^+\,a^+\,a^+\,a^+|\Omega\rangle$$

	Ω_0
0	5
$\frac{1}{2}$	4
1	1

BF symmetry!

- Massless — no central charge. Use lightlike frame.

$$\{Q_\alpha^i, \bar{Q}_{\dot\beta j}\} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta^i_j$$

half the creation & annihilation ops!

$$|\Omega_\lambda\rangle \qquad a^{+i}|\Omega_\lambda\rangle \;\ldots$$

$\boxed{N=1}$

	Ω_{-1}	$\Omega_{-\frac{1}{2}}$	Ω_0	$\Omega_{\frac{1}{2}}$
-1	1			
$-\frac{1}{2}$	1	1		
0		1	1	
$\frac{1}{2}$			1	1
1				1

"Scalar" multiplet. 2 scalars + 1 Weyl fermion. Also massive.
"Chiral" So can give mass supersymmetrically.

"Vector" multiplet + "scalar" multiplet $\Rightarrow$ Same DOF as massive vector multiplet.
Supersymmetric Higgs effect

$\boxed{N=2}$ No central charge.

	Ω_{-1}	$\Omega_{-\frac{1}{2}}$	Ω_0
-1	1		
$-\frac{1}{2}$	2	1	
0	1	2	1
$\frac{1}{2}$		1	2
1		$\times 2$	1

"hyper"

N=2 "scalar" multiplet = 2 N=1 "scalar" multiplets

N=2 "vector" multiplet = 1 N=1 "vector" +
1 N=1 "scalar" multiplet

N=2 massive "vector" multiplet = 1 N=2 massless "vector" + 1 N=2 massless "scalar" multiplets.

- Massive, $N=2$ with central charge

Exercise: Diagonalize $\Rightarrow$

$$\{a_\alpha, a_\beta^+\} = \delta_{\alpha\beta}(M+Z)$$

$$\{b_\alpha, b_\beta^+\} = \delta_{\alpha\beta}(M-Z) \qquad M > Z\,!$$

When $M=Z$, same reps as massive $N=1$.

(BPS)

These are the "short" ~~ireps of~~ $N=2$ susy. They play an essential role in KK reduction, as we shall see....

Fields & Transformations

We'd like to realize the algebra on fields

(Ton of formalism. But useful, as we will see.)

Look at structure of algebra.

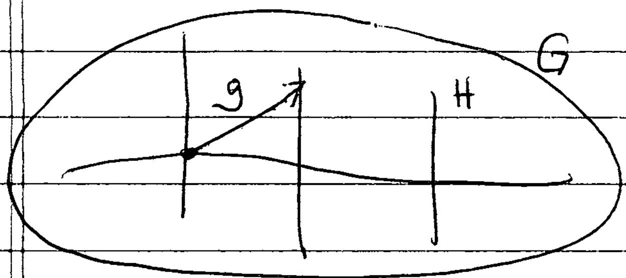
$$[P_a, P_b] = i f_{ab}{}^{c} P_c$$

$$[M, P] \sim P$$

$$[M, M] \sim M.$$

Natural coset structure

$$\frac{\text{SPoincaré}}{\text{Lorentz}} \qquad \frac{\{P, M\}}{\{M\}} \sim \frac{G}{H}$$



identify

$$\Omega = e^{ixP} h$$

One "coordinate" x for each generator P

$$\Omega \rightarrow g\Omega = \Omega' h$$

$$\Rightarrow \quad \Omega' = g \Omega h^{-1}$$

When $g \in H$, we have

$$\Omega' = h \Omega h^{-1}$$

When $g \in G/H$, we have

$$\Omega' = g\Omega$$

In each case, a motion is induced, $x \to x'$

Now introduce a <u>field</u> that transforms in a finite-dimensional rep of H

$\Phi(0)$ $\qquad \Phi(0) \to h \cdot \Phi(0)$

$$= R(h)\,\Phi(0) \qquad h \in H$$

Define $\qquad \Phi(x) = \Omega\, \Phi(0) \qquad \Omega = e^{ix \cdot P}$

$$h \cdot \Phi(x) = h \Omega h^{-1} R(h)\, \Phi(0)$$

$$= \Omega' R(h)\, \Phi(0)$$

$$= R(h)\, \Phi(x') \quad \checkmark$$

$$g \cdot \Phi(x) = g\Omega\, \Phi(0)$$

$$= \Omega' \Phi(0) = \Phi(x') \checkmark$$

Cosets & Covariant Derivatives

Define $\quad \partial_a \Phi(x) = \partial_a \Omega\, \Phi(0)$

$$D_a^L \Phi(x) \equiv iP_a \Omega\, \Phi(0)$$

group transf

$$D_a^R \Phi(x) \equiv \Omega\, iP_a\, \Phi(0)$$

cov. derivative

$$D_a^L = e_a^{Lm} \partial_m$$

$$D_a^R = e_a^{Rm} \partial_m$$

$$D_a^L \Phi(x) = e_a^{Lm} \partial_m \Omega\, \Phi(0) = P_a \Omega\, \Phi(0)$$

$$\Rightarrow \partial_m \Omega = e_m^{La}\, iP_a\, \Omega$$

$$\Rightarrow e_m^{La}\, iP_a = \partial_m \Omega\, \Omega^{-1}$$

Likewise,

$$e_m^{Ra}\, i P_a = \Omega^{-1} \partial_m \Omega$$

with $[P_a, P_b] = i f_{ab}{}^c P_c$ ...

Compute

$$D_a^L D_b^L \Phi(x) = D_a^L P_b \Omega \Phi(0)$$

$$= P_b P_a \Omega \Phi(0)$$

$$\Rightarrow [D_a^L, D_b^L] \Phi(x) = -i f_{ab}{}^c D_c^L \Phi(x)$$

$$[D_a^R, D_b^R] \Phi(x) = +i f_{ab}{}^c D_c^R \Phi(x)$$

$$[D_a^L, D_b^R] \Phi(x) = 0.$$

$\boxed{S_0}$ D_a^R is covariant derivative, obeying group algebra

D_a^L gives the group action on the coordinates, obeying algebra w/ opposite sign.

We have

$$\delta \Phi(x) = \xi^a D_a^L \Phi(x)$$
$$= \xi^a e_a^{Lm} \partial_m \Phi(x)$$

$$D_a \Phi(x) = D_a^R \Phi(x)$$

Covariant derivative because

$$[D_a^L, D_b^R] = 0 \;!$$

Exercise: Show D_a^R is covariant for transformations in H as well as those in G/H

Component Lagrangians

- $N=0$. This is a big hammer!

$$[P_a, P_b] = 0. \qquad e_m^{La} = e_m^{Ra} = \delta_m^a$$

$$D\phi = \partial\phi. \qquad \text{Everything collapses.}$$

Nevertheless:

A Lorentz-invariant Lagrangian, written in terms of fields and their (covariant) derivatives, is automatically Poincaré invariant! $\mathcal{L}(x) \to \mathcal{L}(x')$

$$\mathcal{L} = -\partial_m\phi\,\partial_m\phi^* - m^2\phi\phi^*$$
$$-i\psi\sigma^m\partial_m\bar{\psi} - \frac{m}{2}\psi\psi$$
$$-\frac{m}{2}\bar{\psi}\bar{\psi}$$

- $N=1$ | 1 complex scalar
  1 Weyl spinor.

Same $\mathcal{L}$.

$$\delta\phi = \epsilon\psi$$
$$\delta\psi = 2i\sigma^m\bar{\epsilon}\,\partial_m\phi - m\,\phi^*\epsilon$$

— Automatically Poincaré.
SPoincaré added bonus!

Close algebra:

$$[\delta, \delta]\, \phi$$

$$[\delta, \delta]\, \psi$$

Requires Fermion EOM!
Fermion transf. depends on
mass in and potential!

- $N = 2$

$$\mathcal{L} = -\partial_m \phi^i \partial_m \phi^*_i - m^2 \phi^i \phi^*_i$$

$$- i \psi \sigma^m \partial_m \bar{\psi} - i \chi \sigma^m \partial_m \bar{\chi}$$

$$- m \psi \chi - m \bar{\psi} \bar{\chi}$$

$$\delta \phi^i = \epsilon^i \chi + \bar{\epsilon}^i \bar{\psi}$$

$$\delta \chi = 2i \sigma \bar{\epsilon}_i \partial \phi^i - 2m \phi^*_i \epsilon^i$$

$$\delta \psi = 2i \sigma \bar{\epsilon}_i \partial \phi^{*i} - 2m \phi_i \epsilon^i$$

$N=2$ transformations! $\boxed{\text{CHECK}}$

Hint: i is an $SU(2)$ index

$$\phi^i \qquad \phi^*_i \qquad \phi^{*i} = \epsilon^{ij}\phi^*_j$$

Also, for closure, need Fierz

$$(\psi\phi)\bar{\chi}_{\dot\beta} = -\frac{1}{2}(\phi\sigma^a\bar{\chi})(\psi\sigma^a)_{\dot\beta}$$

References

Wess & Bagger
Buchbinder & Kuzenko
West
Galperin, Ivanov, Ogievetsky, Sokatchev

And for this lecture,

Butter, Ann Phys 325 (2010) 1026

REVIEW

$$i P_a e^{L\ a}_m = \partial_m \Omega\, \Omega^{-1}$$
$$i P_a e^{R\ a}_m = \Omega^{-1} \partial_m \Omega$$

$$D^L_a = e^{Lm}_a \partial_m$$
$$D^R_a = e^{Rm}_a \partial_m$$

$$[P_a, P_b] = i f_{ab}{}^c P_c$$

$$\Rightarrow \quad [D_a^L, D_b^L] = -i f_{ab}{}^c D_c^L$$
$$[D_a^R, D_b^R] = +i f_{ab}{}^c D_c^R$$
$$[D_a^L, D_b^R] = 0$$

$$\delta \Phi(x) = \lambda^a D_a^L \Phi(x)$$ isometry

$$D_a \Phi(x) = D_a^R \Phi(x)$$ covariant derivative.

REVIEW

Massless N=1

Chiral	2 scalars + 1 Weyl	2+2
Vector	1 vector + 1 Weyl	2+2

Massive N=1

Chiral	2 scalars + 1 Weyl	2+2
Vector	1 vector + 2 Weyl + 1 scalar	4+4

Massless N=2

Hyper	4 scalars + 2 Weyls	4+4
Vector	1 vector + 2 Weyls + 2 scalars	4+4

Massive N=2

Long vector	1 vector + 4 Weyls + 5 scalars	8+8
Short vector	1 vector + 2 Weyls + 1 scalar	4+4

Same as massless N=2!

Kaluza Klein!

Short hyper	4 scalars + 2 Weyls	4+4

$N=1$ Superspace

- Coordinates and Isometries
- Covariant Derivatives and Invariant Actions
- $N=1$ Multiplets
  - Chiral
  - Linear
  - Vector
- $N=2$ in terms of $N=1$ superfields
  - Hyper — $D=4$ and
  - Vector — $D=5$!

$N=2$ Superspace

- Harmonic
- Projective ...

Coordinates and Isometries

$$\{ Q_\alpha, \bar{Q}_{\dot\beta} \} = 2\sigma^a_{\alpha\dot\alpha} P_a$$

$$\{ Q_\alpha, Q_\beta \} = 0 \qquad [P, Q_\alpha] = 0$$

$$[P_a, P_b] = 0$$

$$\Omega = e^{i x \cdot P} = e^{i(xP + \theta Q + \bar\theta \bar Q)}$$

$$D^L_A = e_A^{L\,M} \partial_M \qquad A = (a, \alpha, \dot\alpha)$$

$$m = (m, \mu, \dot\mu)$$

$$D^R_A = e_A^{R\,M} \partial_M$$

$$e_m^{L\,A}\, iP_A = \partial_m \Omega\, \Omega^{-1}$$

$$e_m^{R\,A}\, iP_A = \Omega^{-1} \partial_m \Omega$$

- Compute! Helped by fact that

$$[P, \{Q, Q\}] = 0$$

etc...

- Find:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\,\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_a$$

$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^a_{\alpha\dot\alpha}\partial_a$$

$$\{D_\alpha, D_\beta\} = 0 \qquad \{D_\alpha, \bar D_{\dot\alpha}\} = -2i\,\sigma^a_{\alpha\dot\alpha}\partial_a$$

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,\sigma^a_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_a$$

$$\bar Q_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + i\theta^\alpha\sigma^a_{\alpha\dot\alpha}\partial_a$$

$$\{Q_\alpha, Q_\beta\} = 0 \qquad \{Q_\alpha, \bar Q_{\dot\alpha}\} = +2i\,\sigma^a_{\alpha\dot\alpha}\partial_a$$

$$D_a = P_a = -i\frac{\partial}{\partial x^a}$$

- All act on superfields

$$\Phi(x,\theta,\bar\theta) = \Omega\,\Phi(0)$$
$$= e^{i(xP + \theta Q + \bar\theta\bar Q)}\,\Phi(0)$$

$$\delta\Phi = i(\epsilon Q + \bar\epsilon\bar Q)\,\Phi(x,\theta,\bar\theta)$$

$\uparrow$ Differential Operator!

- Since $\theta, \bar{\theta}$ are Grassmann, the power series cuts off.

$$\Phi(x,\theta,\bar{\theta}) = \phi(x) + \theta\chi(x) + \bar{\theta}\bar{\psi}(x)$$
$$+ \theta\theta F(x) + \cdots$$

Finite number of component fields, all related by SUSY!

The problem is that the reps are too large. Need to reduce number through covariant constraints.

Find off-shell reps, some DOF propagating, some auxiliary.

• Invariant Action

$$S = \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \mathcal{L}(x,\theta,\bar{\theta})$$

$$= \int d^4x \, D^2 \bar{D}^2 \mathcal{L}(x,\theta,\bar{\theta}) \Big|$$

Check invariance:

$$Q_\alpha = D_\alpha - 2i \, \sigma^a_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_a$$

$$\bar{Q}_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}} + 2i \, \theta^\alpha \sigma^a_{\alpha\dot{\alpha}} \partial_a$$

$$\delta S = \int d^4x \, D^2 \bar{D}^2 (\epsilon Q + \bar{\epsilon}\bar{Q}) \mathcal{L} \Big|$$

$$= \int d^4x \, D^2 \bar{D}^2 (\epsilon D + \bar{\epsilon} \bar{D}) \mathcal{L} \Big| \quad \text{up to } \partial$$

$$= \int d^4x \, D^2 \epsilon D \, \bar{D}^2 \mathcal{L} \Big| + cc \quad \text{up to } \partial$$

$$= 0!$$

<u>Automatically</u> $N=1$ supersymmetric!

N=1 Multiplets

- Chiral (or scalar) Multiplet

Canonical example.

Covariant constraint

$$\bar{D}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0$$

- Consistent $\{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$
- Covariant $\{\bar{D}_{\dot\alpha}, Q_\alpha\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$

Solution:

$$\Phi(x,\theta,\bar\theta) = \Phi(x - i\theta\sigma\bar\theta, \theta, 0)$$

$\Longrightarrow$

$$\Phi(x,\theta,\bar\theta) = \varphi(x) + \theta\psi(x) + \theta\theta F(x) + \cdots$$

dim 1	dim $\frac{3}{2}$	dim 2
2 bosons	4 fermions	2 bosons

N=1 scalar multiplet!

The susy transformations are the isometries generated by Q & $\bar{Q}$

$$\delta \Phi(x,\theta,\bar{\theta}) = i\left(\epsilon Q + \bar{\epsilon}\bar{Q}\right)\Phi(x,\theta,\bar{\theta})$$

$\Rightarrow$

$$\delta \phi = \epsilon \chi$$

$$\delta \psi = 2i\,\sigma^{a}\bar{\epsilon}\,\partial_a \phi + 2\epsilon F$$

$$\delta F = i\,\bar{\epsilon}\bar{\sigma}^{a}\partial_a \chi$$

These transformations close off shell, ie w/o EOM.

4+4 DoF

One complex auxiliary field F.

The action is just

$$S = \int d^4x \, D^2 \bar{D}^2 \, \bar{\Phi} \, \Phi \Big|$$

$$+ \int d^4x \, D^2 \, \tfrac{1}{2} m \, \Phi^2 \Big|$$

$$= \int d^4x \; -\partial\phi \, \partial\phi^* - i \psi \sigma^a \partial_a \bar{\psi} + F^* F$$

$$+ \left[m \phi F - \tfrac{m}{2} \psi^2 + \text{h.c.} \right]$$

F EOM is a constraint

$$F = -m \phi^*$$

Note δF vanishes using ψ EOM

So all is well...

In fact, for chiral multiplet

$$S = \int d^4x \, D^2 W(\Phi) \Big|$$

is also invariant.

$$\delta S = \int d^4x \, D^2 (\epsilon Q + \bar{\epsilon}\bar{Q}) W \Big|$$

$$= \int d^4x \, D^2 (\epsilon D + \bar{\epsilon}\bar{D}) W \Big| \quad \text{up to } \partial$$

$$= 0$$

$W(\Phi)$ is an analytic function called the super potential.

$$V = \left| \frac{\partial W}{\partial \phi} \right|^2$$

mass & Yukawas $\quad \dfrac{\partial^2 W}{\partial \phi \, \partial \phi}$

- Linear Multiplet (a tensor)

In D=4, a two-form gauge field also describes one DOF.

$$H_{[abc]} = \partial_{[a} B_{bc]}$$

$$\delta B_{[ab]} = \partial_{[a} \Lambda_{b]} \quad \text{invariance}$$

Bianchi:

$$\epsilon^{abcd} \partial_a H_{[bcd]} = 0$$

EOM

$$\partial^a H_{[abc]} = 0$$

Duality $\Rightarrow$

$$H_{[abc]} = \epsilon_{abcd} \partial^d \phi$$

H EOM $\Rightarrow$ $\partial\phi$ Bianchi
H Bianchi $\Rightarrow$ ϕ EOM

unconstrained

$$S = \int d^4x \left[H^2 + \epsilon^{abcd} H_{abc} \partial_d \phi \right]$$

ϕ EOM $\Rightarrow$ H Bianchi

H EOM $\Rightarrow$ $H = \epsilon \partial \phi$

$$\Rightarrow S \sim \int d^4x \, -\partial\phi\partial\phi$$

All this holds supersymmetrically as well

- In Superspace, define

$$\Gamma = \bar{\Gamma}$$

$$D^2 \Gamma = 0$$

$$\Rightarrow \Gamma = A(x) + \theta \chi(x) + \theta \sigma^a \bar{\theta} V_a$$

with $\partial_a V^a = 0 \Rightarrow V^a = \epsilon^{abcd} H_{bcd}$

Note: DOF balance off-shell!

Duality:

$$S = \int d^4x \, D^2 \bar{D}^2 \left(-\tfrac{1}{2}\Gamma^2 + \Gamma(\Phi + \bar{\Phi}) \right) \Big|$$

where Γ is unconstrained and Φ is chiral, $\bar{D}_{\dot{\alpha}} \Phi = 0$

Vary Φ: $\bar{D}_{\dot{\alpha}} \Phi = 0 \implies$

$$\delta \Phi = \bar{D}^2 \delta \Psi$$

$$\implies D^2 \Gamma = 0$$

Vary Γ: $\Gamma = \Phi + \bar{\Phi}$

Plug in to find

$$S = \int d^4x \, D^2 \bar{D}^2 \, \bar{\Phi} \Phi \Big|$$

- Vector Multiplet

$V = \overline{V}$

V is undefined up to a shift,

$V \to V - \Lambda - \bar{\Lambda}$ $\qquad \bar{D}_{\dot{\alpha}} \Lambda = 0$

"gauge transformation" (abelian)

Use this freedom to write V in Wess Zumino gauge,

$$V = \theta \sigma^a \bar{\theta} A_a + \theta\theta \bar{\theta} \bar{\lambda} + \bar{\theta}\bar{\theta} \theta \lambda$$

$$+ \theta\theta \bar{\theta}\bar{\theta} D$$

There is a residual gauge symmetry $A_m \to A_m - \partial_m \lambda$

4+4 off-shell DOF.

$$\delta A_a = i \left(\epsilon \sigma_a \bar{\lambda} + \bar{\epsilon} \bar{\sigma}_a \lambda \right)$$

$$\delta \lambda = i \epsilon D + \sigma^{ab} \epsilon F_{ab}$$

$$\delta D = \bar{\epsilon} \bar{\sigma}^a \partial_a \lambda - \epsilon \sigma^a \partial_a \bar{\lambda} .$$

one real auxiliary.

$W_\alpha = \frac{1}{4}\bar{D}^2 D_\alpha V$ is chiral

and gauge invariant.

$$W_\alpha \to \frac{1}{4}\bar{D}^2 D_\alpha (V + \Lambda + \bar{\Lambda})$$

$$= \frac{1}{4}\bar{D}^2 D_\alpha V \qquad \text{using } \{D,\bar{D}\} = \partial$$

$$W_\alpha = \lambda_\alpha + D\,\theta_\alpha + \sigma^{mn}{}_\alpha{}^\beta \theta_\beta F_{mn}$$

$$+ \theta^2 \sigma^a_{\alpha\dot{\alpha}} \partial_a \bar{\lambda}^{\dot{\alpha}}$$

Lagrangian:

$$S = \int d^4x\, D^2\, W^\alpha W_\alpha \Big|$$

$$+ \int d^4x\, D^2 \bar{D}^2\, \bar{\Phi}\, e^V\, \Phi$$

$$+ \int d^4x\, D^2\, W(\Phi) \Big| \quad \leftarrow \text{gauge inv't}$$

$$\Phi \to e^\Lambda \Phi$$

N=1 YM coupled to matter!

- $\underline{N=2}$ in terms of $\underline{N=1}$ Superfields

① Hyper $\qquad i=1,2$

$$\int d^4x\, D^2\bar{D}^2\, \bar{\Phi}_i\, \Phi^i \Big|$$

$$+ \int d^4x\, D^2\, m\, \delta_{ij}\, \Phi^i\, \Phi^j \Big|$$

$N=1$ manifest.

$N=2$ requires another supersymmetry.

$$\delta\Phi^i = \epsilon^{ij}\, \bar{D}\bar{D}\left(\theta\eta + \bar{\theta}\bar{\eta}\right)\bar{\Phi}_j$$

$$+ m\, \Phi^i\, \eta\theta$$

preserves chirality

Closes into $N=2$ with $Z=m$, as required. Need superfield EOM:

$$D^2\Phi^i = m\,\bar{\Phi}^i$$

② Vector

$$\int d^4x\, D^2 WW \Big| \quad + \quad \int d^4x\, D^2 \bar{D}^2\, \bar{\Phi}\, \Phi$$

Abelian!,

$$\delta \Phi = \eta W$$

$$\delta V = (\Phi + \bar{\Phi})(\theta\eta + \bar{\theta}\bar{\eta})$$

up to gauge

It is remarkable that this set-up has the full complement of $N=2$ auxiliary fields!

$$F, D \iff D^{(ij)} \qquad 8+8 \text{ DOF}$$

So it even closes off-shell!

- $\underline{D=5}$ in terms of $\underline{N=1}$ $\underline{\text{Superfields}}$

① Hyper

$$\int d^4x\, D^2\bar{D}^2\, \bar{\Phi}_i \Phi^i \Big|$$

$$+ \int d^4x\, D^2\, \Phi^i \left(\epsilon_{ij}\, i\partial_5 + \delta_{ij}\, m\right)\Phi^j \Big|$$

$$\delta\Phi^i = \epsilon^{ij}\, \bar{D}\bar{D}\left(\theta\eta + \bar{\theta}\bar{\eta}\right)\bar{\Phi}_j$$

$$+ \left(m\epsilon^{ij} + \delta^i_j\, i\partial_5\right)\Phi^j\, \theta\eta$$

∂_5 acts like a mass — KK!

② Vector

$$\int d^4x\, D^2\, WW\Big| + \int d^4x\, D^2\bar{D}^2\left(\Phi + \bar{\Phi} - \partial_5 V\right)^2$$

Invariant under $\Phi \to \Phi + \partial_5\Lambda$

$V \to V + \Lambda + \bar{\Lambda}$

$$\delta \Phi = \eta W$$

$$\delta V = (\Phi + \bar{\Phi} - \partial V)(\theta\eta + \bar{\theta}\bar{\eta})$$

up to gauge

Exercise — KK reduce this!

See Hull, Karlhede, Lindström, Roček
Nucl. Phys. B266 (1986) 1

Arkani-Hamed, Gregoire, Wacker
JHEP 0203 (2002) 055.

$N=2$ Superspace (Brute Force)

$$\{D_\alpha^i, \bar{D}_{\dot\alpha j}\} = 2\delta^i_j\, \sigma^a_{\alpha\dot\alpha} P_a$$

$$\{D_\alpha^i, D_\beta^j\} = \epsilon_{\alpha\beta}\epsilon^{ij} Z$$

coordinates $x^a,\ \theta_\alpha^i,\ \bar\theta_{\dot\alpha j}$

$\Phi(x, \theta_\alpha^i, \bar\theta_{\dot\alpha j})$ HUGE!

Constraint?

- Try chiral $\bar{D}_{\dot\alpha i}\Phi = 0$

(only consistent for $Z=0$)

$$\Phi = A + \theta^i\psi_i + \theta^i\sigma^{ab}\theta^j\epsilon_{ij}F_{ab}$$

$$+ \theta^i\theta^j D_{(ij)} + \text{auxiliaries!}$$

16+16 DOF

Reality? Bianchi?

Impose $\Box\Phi = (\bar{D}^1)^2(\bar{D}^2)^2\bar\Phi$!

Invariant Action

$$S = \int d^4x \, (D^1)^2 (D^2)^2 \, \Phi^2 \Big|$$

chiral !

We get YM field from chiral constraint !

- How to get hypers?

$$\Phi^i \Big| = \phi^i$$

$$D_\alpha^{(i} \Phi^{j)} = 0 = \bar{D}_{\dot\alpha}^{(i} \Phi^{j)}$$

Consistent with D-algebra — check.

$$\Rightarrow \quad \Phi = \phi^i + \theta^i \psi + \bar\theta^i \bar\chi + \ldots \text{ derivs.}$$

No auxiliary fields. Derivative terms impose EOM

Too strong....

Theorem

For $N > 1$, certain off-shell reps require an infinite number of auxiliary fields.

$N=2$ $D=4$ SYM OK

$N=2$ $D=4$ Hyper ∞

- Sketch of proof for hypers.

Consider auxiliary fermions. Parity $\Rightarrow$ Dirac.

$$\mathcal{L} = \int d^4x \; \chi \psi \qquad [\chi] = \frac{3}{2} \quad [\psi] = \frac{5}{2}$$

Let there be F such pairs. Then we have

$$(\underset{\substack{\uparrow \\ \text{aux}}}{2F} + \underset{\substack{\uparrow \\ \text{physical}}}{1})\, 8 \quad \text{fermionic DoF}$$

On the other hand, fields form reps of

$$\{ Q_\alpha^i, \bar{Q}_{\dot\beta j} \} = 2\delta^i_j \, \sigma_{\dot\beta\alpha}{}^a P_a$$

2^4 fields.

How many fermionic DOF?

$$\tfrac{1}{2} \cdot 2^4 \cdot 2\,(2J+1)(2S+1)$$

complex fields

Always a multiple of 16.

$16F+8$ is only a multiple of 16 as $F \to \infty$.

Infinite number of auxiliaries!

[Compare SYM. Real bosonic fields. Finite number of auxiliaries.]

Way out $\Rightarrow$ ∞ auxiliary fields.

For $N=2$, exploit $SU(2)$ automorphism group

Coset: $\dfrac{N=2}{\text{Lorentz}} \times \dfrac{SU(2)}{U(1)}$

$\uparrow$ sphere.

$N=2$ superspace is regular $N=2$ superspace $\times S^2$

$\Phi(x, \theta^i_\alpha, \bar\theta_{\dot\alpha j}, \text{sphere})$ How to describe?

- Harmonic superspace – expand in spherical harmonics
- Projective superspace – use projective coordinates.

We'll look at both.

Harmonic Superspace

In harmonic superspace, expand superfields in spherical harmonics.

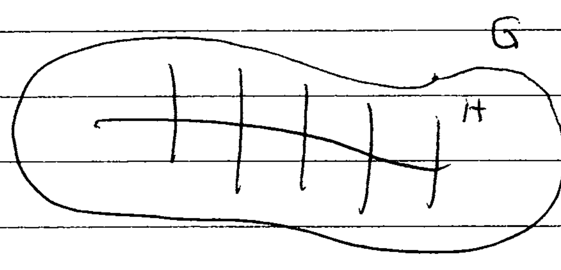
One harmonic superfield $\Longrightarrow$ ∞ number of regular superfields.

Each harmonic coefficient is itself a regular superfield.

SU(2) automorphism groups acts appropriately on the harmonics. (SU(2) is isometry group of sphere)

First, harmonics. To construct harmonics, again use coset approach.

$$\underset{SU(2)}{G} = \underset{\text{coset}}{\Omega}\,\underset{U(1)}{H}$$



Pick basis

$$T^{++} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T^{--} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$T^{0} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

To project to sphere, must be $U(1)$ invt!

$$[T^0, T^{\pm\pm}] = \pm 2\, T^{\pm\pm}$$

$$[T^{++}, T^{--}] = T^0$$

$$G = e^{i(\xi T^{++} + \bar{\xi} T^{--})}\, e^{i\beta T^0}$$

$$= \left(\begin{array}{c|c} u_i^+ & u_i^- \end{array} \right) e^{i\beta T^0}$$

$$G^\dagger = G^{-1} \Rightarrow \quad (u_i^+)^* = u^{i-}$$

$$(u_i^-)^* = u^{i+}$$

$$\epsilon^{ij} u_i^+ u_j^- = 1 \qquad 8-6 = 2 \text{ DOF} \checkmark$$

In this approach, functions of well-defined U(1) charge project to functions on the sphere

$$f^{+}(u,\bar{u}) = f^{i} u_i^{+} + \underset{\swarrow\text{ spin}}{f^{(ijk)}} \underset{\swarrow\text{ spherical harmonics}}{u_i^{+} u_j^{+} u_k^{-}} + f^{(ijklm)} u_i^{+} u_j^{+} u_k^{+} u_l^{-} u_m^{-} + \cdots$$

We can also construct appropriate covariant derivatives:

$$\Omega^{-1} d\Omega = i w^{++} T^{--} + i w^{--} T^{++} + i w^{0} T^{0}$$

$$w^{\pm\pm} = \mp i\, u^{\pm j}\, du_j^{\pm}$$

$$w^{0} = i\, u^{+j} du_j^{-} = i\, u^{-j} du_j^{+}$$

$\Rightarrow$

$$D^{++} = u^{+i} \frac{\partial}{\partial u^{-i}} \qquad D^{--} = u^{-i} \frac{\partial}{\partial u^{+i}}$$

$$D^{0} = u^{+i} \frac{\partial}{\partial u^{+i}} - u^{-i} \frac{\partial}{\partial u^{-i}}$$

The D's obey the SU(2) algebra

$$[D^0, D^{\pm\pm}] = \pm 2 D^{\pm\pm}$$

$$[D^{++}, D^{--}] = D^0.$$

Check $\quad D^0 \, (u^{+i} u_i^- = 1) \quad \checkmark$

$\quad D^{\pm\pm} \, (u^{+i} u_i^- = 1) \quad \checkmark$

Acting on a function of well-defined charge

$$D^0 f^{(q)} = q \, f^{(q)}$$

<u>Why</u> <u>do</u> <u>we</u> <u>care</u>??

Use $u_i^{\pm}$ to rewrite SUSY algebra...

$\partial_5 + i\partial_6$

$$\{D_\alpha^i, D_\beta^j\} = \epsilon_{\alpha\beta}\,\epsilon^{ij}\, Z$$

$$\{D_\alpha^i, \bar{D}_{\dot\beta}^j\} = \epsilon^{ij}\,\sigma^a_{\alpha\dot\beta}\, P_a$$

$\Longrightarrow$

$$\{D_\alpha^+, D_\beta^+\} = 0 \qquad \{D_\alpha^-, D_\beta^-\} = 0$$

$$\{D_\alpha^+, \bar{D}_{\dot\beta}^+\} = 0 \qquad \{D_\alpha^-, \bar{D}_{\dot\beta}^-\} = 0$$

$$\{D_\alpha^\pm, \bar{D}_{\dot\alpha}^\mp\} = \sigma^a_{\alpha\dot\alpha} P_a$$

$$[D^{++}, D_\alpha^+] = 0$$

$$[D^{++}, D_\alpha^-] = D_\alpha^+ \qquad \text{etc.}$$

$\Longrightarrow$

Therefore it is consistent to impose

$$D_\alpha^+ \Phi = 0 \qquad \bar{D}_{\dot\alpha}^+ \Phi = 0$$

Analytic subspace of N=1 !

The analytic subspace is like the chiral subspace

$$\bar{D}_{\dot\alpha} \Phi = 0 \ldots$$

Solution:

$$y = x - 2i \, \theta^{(i} \sigma \bar{\theta}^{j)} u_i^+ u_j^-$$

$$\theta^{\pm}_{\alpha} = u_i^{\pm} \theta^i_{\alpha}$$

$$\bar{\theta}^{\pm}_{\dot\alpha} = u_i^{\pm} \bar{\theta}^i_{\dot\alpha}$$

[In this basis the explicit forms of D^{++}, D^{--}, D^0 change, but the algebra does not...]

Now consider an analytic field

$$\Phi^+ = \Phi^i u_i^+ + \Phi^{(ijk)} u_i^+ u_j^+ u_k^- + \cdots$$

For fun, impose:

$$D^{++}\Phi^{+} = 0$$

$$\Rightarrow \quad \Phi^{+} = \Phi^{i} u_i^{+}$$

$$D_\alpha^{+}\Phi^{+} = D_\alpha^{i}\Phi^{j} u_i^{+}u_j^{+} = 0$$

$$\bar{D}_{\dot\alpha}^{+}\Phi^{+} = \bar{D}_{\dot\alpha}^{i}\Phi^{j} u_i^{+}u_j^{+} = 0$$

$$\Rightarrow \quad D_\alpha^{(i}\Phi^{j)} = 0$$

$$\bar{D}_{\dot\alpha}^{(i}\Phi^{j)} = 0$$

So <u>EOM</u> is $D^{++}\Phi^{+} = 0$

To find an <u>action</u>, we need <u>integration</u> over <u>harmonics</u>:

$$\int du = 1 \qquad \int du\, uuu = 0.$$

$$S = \int d^4x \int du\, D^- D^- \bar{D}^- \bar{D}^- \mathcal{L}^{(+4)}(\Phi, u)$$

$\mathcal{L}^{(+4)}$ needs to be "real" and analytic, of charge +4.

Φ^+ is in general complex. What's its conjugate?

Use antipodal map on sphere:

$$(u^{+i})^\times = -u_i^+ \qquad (u_i^-)^\times = u^{-i}$$

$$\left(\epsilon^{ij} u_i^+ u_j^- = 1\right)^\times \qquad \text{invariant!}$$

Define conjugation as $\times$ on harmonics, $*$ on numbers and fields. denote by $\vee$.

Preserves analytic subspace.

$$S = \int d^4x \int du\, D^- D^- \bar{D}^- \bar{D}^- \,\breve{\Phi}^+ D^{++} \Phi^+$$

<u>EOM</u> $\qquad D^{++}\Phi^{+} = 0$ ✓

<u>Aside</u>: How to include mass? Use extra dimensions.

In analytic basis,

$$D_A^{++} = D^{++} - 2i\,\theta^{+}\sigma^{a}\bar{\theta}^{+}\partial_a + \theta^{+}\frac{\partial}{\partial\theta^{-}} + \bar{\theta}^{+}\frac{\partial}{\partial\bar{\theta}^{-}}$$

$$+ i\,\theta^{+2}\frac{\partial}{\partial z} - i\left(\bar{\theta}^{+}\right)^{2}\frac{\partial}{\partial\bar{z}}$$

Two extra dimensions....
Complex mass!

What about <u>gauge fields</u>?

$$\Phi^{+} \rightarrow e^{i\lambda}\Phi^{+}$$

$$D_{\alpha}^{+}\lambda = \bar{D}_{\dot{\alpha}}^{+}\lambda = 0.$$

preserve analytic subspace

promote $D^{++} \to D^{++} + i V^{++}$

with V^{++} analytic. Under a gauge transformation,

$$V^{++} \to V^{++} - D^{++}\lambda$$

Let's use λ to fix WZ gauge

$$\lambda = \lambda + \lambda^{(ij)} u_i^+ u_j^- + \ldots$$

$$D^{++}\lambda = \lambda^{(ij)} u_i^+ u_j^+ + \ldots$$

$$V^{++} = V^{(ij)} u_i^+ u_j^+ + \ldots$$

So we can gauge away "lowest" component in θ. Looking to higher components, we find

$$V^{++}_{WZ} = \theta^+ \sigma^a \bar{\theta}^+ A_a + \theta^{+2} \phi^* + \bar{\theta}^{+2} \phi$$

$$+ \bar{\theta}^{+2} \theta^+ \psi^i u_i^- + \theta^{+2} \bar{\theta}^+ \bar{\psi}^i u_i^-$$

$$+ \theta^{+2} \bar{\theta}^{+2} D^{ij} u_i^- u_j^-$$

SYM DOF!

And the action

$$\int d^4x \int du \; D^{-2}\bar{D}^{-2} \; \bar{\Phi}^{+} (D^{++} + iV^{++}) \Phi^{+}$$

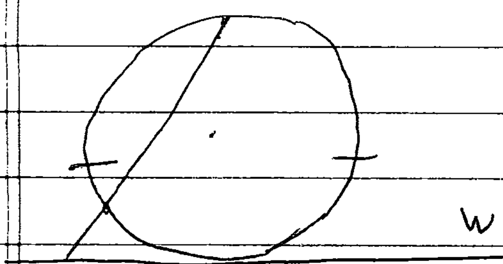
The gauge-field KE is not as easy... It is nonlocal in the harmonics...

Projective Superspace

Variant of Harmonic Superspace

- SU(2) not manifest
- Closer connection to $N=1$ superfields

Not identical, but close....



w covers south chart

$z = 1/w$ covers north

SU(2) isometries realized nonlinearly

$$\delta z = 1 + z^2 \qquad \delta z = i(1 - z^2)$$

$$\delta z = iz$$

In projective superspace, generic superfields are expressed in power series in w, not $\bar{w}$.

$$\Xi = \sum_{n=-\infty}^{\infty} \Xi_n(x, \theta, \bar\theta)\, w^n$$

where

$$\nabla_\alpha \Xi = \bar\nabla_{\dot\alpha} \Xi = 0$$

and

$$\nabla_\alpha = w\, D^1_\alpha - D^2_\alpha$$
$$\bar\nabla_{\dot\alpha} = \bar D_{\dot\alpha} + w \bar D_\alpha{}^2$$

$$\{\nabla_\alpha, \bar\nabla_{\dot\alpha}\} = 0 \qquad \{\nabla_\alpha, \nabla_\beta\} = 0$$

(for $Z = 0$)

Projective superfields obey, therefore

$$D^2_\alpha \Xi_n = D^1_\alpha \Xi_{n-1}$$

$$\bar D_{\dot\alpha 2} \Xi_n = -\bar D_{\dot\alpha 1} \Xi_{n+1}$$

This allows us to write Ξ_n entirely in terms of θ^1_α, $\bar\theta_{\dot\alpha 1} = \theta_\alpha, \bar\theta_{\dot\alpha}$.

$N=1$ superfields!

Likewise, $D_\alpha^{\,1} = D_\alpha$

$$\bar{D}^{\dot\alpha}_{1} = \bar{D}^{\dot\alpha}$$

It is not hard to see that the conjugation operation — antipodal mapping — is defined by

$$\check{\Xi} = \sum_n (-1)^n \bar{\Xi}_{-n}\, w^n$$

We will see that hypers coupled to gauge fields are described by superfields

$$\Upsilon = \sum_{n=0}^{\infty} \Upsilon_n w^n$$ holomorphic away from North pole

$$\check{\Upsilon} = \sum_{n=0}^{\infty} (-)^n \bar{\Upsilon}_n \frac{1}{w^n}$$ holomorphic away from South pole

$$V = \check{V} = \sum_{n=-\infty}^{\infty} V_0\, w^n$$

Arctic, Antarctic & tropical multiplets ...

Let the action be tropical

$$\mathcal{L} = \check{\mathcal{L}}$$

Solving the constraint,

$$\mathcal{L}_n = (-)^n \bar{\mathcal{L}}_{-n}$$

$\Rightarrow$ $\mathcal{L}_0$ real

$$S = \int d^4x \, D^2 \bar{D}^2 \mathcal{L}_0 \Big|$$

$$= \frac{1}{2\pi i} \int d^4x \oint \frac{dw}{w} \, D^2 \bar{D}^2 \mathcal{L} \Big|$$

Invariant?

$$\delta S = \int d^4x \left(\epsilon_i Q^i + \bar{\epsilon}^i \bar{Q}_i \right) D^2 \bar{D}^2 \mathcal{L}_0 \Big|$$

$$= \int d^4x \left(\epsilon_i D^i + \bar{\epsilon}^i \bar{D}_i \right) D^2 \bar{D}^2 \mathcal{L}_0 \Big|$$

$$= \int d^4x \left(\epsilon_2 D^2 + \bar{\epsilon}^2 \bar{D}_2 \right) D^2 \bar{D}^2 \mathcal{L}_0 \Big|$$

$$= \int d^4x \, D^2 \bar{D}^2 \left(\epsilon_2 D^1 \mathcal{L}_{-1} \Big| - \epsilon^2 \bar{D}_1 \mathcal{L}_1 \Big| \right)$$

$$= 0$$

Claim:

Gauge-invariant action

$$S = \frac{1}{2\pi i} \int d^4x \int \frac{dw}{w} D^2 \bar{D}^2 \check{\Upsilon} e^{V} \Upsilon$$

$\mathcal{L} = \check{\mathcal{L}}$.

Under gauge, $\Upsilon \to e^{\Lambda} \Upsilon$ arctic

$V \to V - \Lambda - \check{\Lambda}$

Perfect.

Let's look closer, though...

V and Λ are arctic $\Upsilon_{-n} = 0, \; n > 1$.

$\Rightarrow \bar{D}_{\dot{\alpha}} \Upsilon_0 = 0 \qquad \bar{D}_{\dot{\alpha}} \Lambda_0 = 0$

$\Rightarrow \bar{D}^2 \Upsilon_1 = 0 \qquad \bar{D}^2 \Lambda_1 = 0$

Chiral plus Linear multiplets.

Thus we can go to WZ gauge, and gauge away <u>all</u> components of V_1 except for V_0 & V_1

$$V_0 \Rightarrow V_0 - \Lambda_0 - \bar{\Lambda}_0 \qquad \bar{D}_{\dot{\alpha}} \Lambda_0 = 0$$

$$V_1 \rightarrow V_1 - \Lambda_1 - \bar{\Lambda}_1 \qquad \bar{D}^2 \Lambda_1 = 0$$

Also, in WZ gauge,

$$V_0 = \theta \sigma \bar{\theta} + \cdots$$
$$V_1 = \theta^2 + \cdots$$

$$\Rightarrow V_0 V_1 = 0 \qquad V_1^{\,2} = 0$$

To find the <u>N=1</u> <u>Action</u>, write, then

$$V = V_0 + w V_1 + \frac{1}{w} V_{-1} + \cdots$$

$$\Upsilon = \Phi + w \Gamma + \cdots \qquad \bar{D}_{\dot{\alpha}} \Phi = 0$$

$$\bar{D}^2 \Gamma = 0$$

auxiliaries

Find

$$S = \int d^4x\, D^2 \bar{D}^2 \Big[\bar{\Phi} e^{V_0} \Phi - \bar{\Gamma} e^{V_0} \Gamma - \bar{\Gamma} V_1 \Phi + \Gamma V_1 \bar{\Phi} \Big] \Big|$$

Exercise: Convince yourself that in WZ gauge, V_1 has a chiral multiplet's worth of fields!

Exercise: Generalize this to include central charges! (Hint: go to D=5, 6)

Connect to Harmonic Superspace:

$u^{+i} = u^{+1} w^i \qquad w^i = (1, w)$

$u_i^- = (u^{+1})^* \bar{w}_i \qquad \bar{w}_i = (1, \bar{w})$

$|u^{+1}|^2 = \dfrac{1}{1 + w\bar{w}}$

Then

$$D_\alpha^+ = -u^{+1} \nabla_\alpha \qquad \bar{D}_{\dot\alpha}^+ = -u^{+1} \bar{\nabla}_{\dot\alpha}$$

$$D^{++} = (u^{+1})^2 (1 + w\bar{w}) \frac{\partial}{\partial \bar{w}}$$

$$D_\alpha^+ \Phi = \bar{D}_{\dot\alpha}^+ \Phi = 0$$ projective superfield.

$$D^{++} \Phi^+ = 0 \Rightarrow \Phi^+$$ holomorphic (away from poles...)

Can pass from harmonic action to projective action by regulating poles at poles, with

$$\Phi^+ = u^{+1} \Upsilon(w) \quad \ldots$$

see afternoon session.

Additional references, this lecture.

Rivelles & Taylor, Phys Lett 121 B (1983) 37

Lindström & Roček, Comm Math Phys 128 (1990) 191
Kuzenko, Int J Mod Phys A14 (1999) 1737.
Kuzenko & Linch, JHEP 02 (2006) 038,
Linch, U Maryland PhD thesis.

<u>Exercise</u>: Have at it!

(C4)

PM session

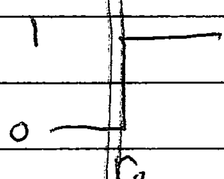
$$\frac{1}{2\pi} \int dw\, d\bar{w} \; \frac{1}{(1+w\bar{w})^2} \; f(w)$$

$$= \frac{-1}{2\pi} \int dw\, d\bar{w} \; \frac{1}{w} \frac{\partial}{\partial \bar{w}} \frac{1}{1+w\bar{w}} \; f(w)$$

$$= \frac{1}{2\pi} \int dw\, d\bar{w} \; \frac{1}{1+w\bar{w}} \frac{\partial}{\partial \bar{w}} \frac{f(w)}{w}$$

regulate.

$$= \frac{1}{2\pi} \int d\theta \int r\, dr \; \frac{1}{1+r^2} \frac{\partial}{\partial \bar{w}} \theta(r-r_0) \frac{f(w)}{w}$$



$$= \int dr \; \frac{1}{1+r^2} \left[\frac{\partial}{\partial r} \theta(r-r_0) \right] f_0$$

$$= \frac{1}{1+r_0^2} f_0 = f_0$$

$$= \frac{1}{2\pi i} \oint dw \; \frac{f(w)}{w}.$$

done.