Preliminary problems

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1 Meromorphic differentials on the sphere

We can describe the sphere, as a complex manifold, by a stereographic projection: points away from the North pole are labeled by a complex coordinate z, and the region $z \to \infty$ around the North pole is mapped to the origin by a coordinate transformation $z' = 1/z$.

A degree k meromorphic differential on the sphere is an expression $\phi_k(z) = F(z)dz^k$, where $F(z)$ is a rational function on the complex plane. The behavior of the differential around the point at infinity becomes manifest in the z' coordinates $\phi_k(z') = F(1/z')(-dz'/(z')^2)^k$. We will be interested at first in degree k differentials which have a pole of order k at several points z_i, which may include ∞. In other words, we allow

$$\phi_k \sim \frac{\alpha_i}{(z-z_i)^k} dz^k + \cdots \tag{1}$$

The coefficient α_i is called the residue of the differential at z_i.

We are typically interested in properties of such objects up to fractional linear transformations $z \to \frac{az+b}{cz+d}$, $ad - bc = 1$.

i) Warmup: Show that the residues of $\phi_k(z)$ are invariant under fractional linear transformations. The points z_i of course transform as $z_i \to \frac{az_i+b}{cz_i+d}$.

ii) Counting parameters. Consider the space of all degree k meromorphic differentials with fixed residues α_i at n points z_i. What is the dimension $d_{k,n}$ of that space?

2 Decoupling limit

Consider a setup with $n+m$ points z_i on the sphere, where m of the points are very close to the origin: $|z_i| < |q|$ if $i = 1 \cdots m$ for a small complex parameter q. The other n points are at generic locations, say $|z_i| > 1$ if $i > m$. We can simply rescale $z = q\tilde{z}$ to get to a different description where n points are close to $\tilde{z} = \infty$ and m at generic locations $|\tilde{z}_i| < 1$. Consider a family of degree k differentials $\phi_k(q)$ with poles at the z_i. It is rather natural to consider a family for which $\phi_k(q, z) \to \phi_k^{(1)}(z)$ as $q \to 0$ with z fixed, where $\phi_k^{(1)}(z)$ is a degree k differential with degree k poles at $n+1$ points 0 and z_i, $i > m$. On the other hand, if we take the limit at fixed $\tilde{z}$ coordinate, it is natural to require $\phi_k(q, \tilde{z}) \to \phi_k^{(2)}(\tilde{z})$ where $\phi_k^{(2)}(\tilde{z})$ is a degree k differential with degree k poles at $m+1$ points ∞ and $\tilde{z}_i$, $i < m+1$.

i) Build a few examples of such families for small values of k, m, n.

ii) Can $\phi_k^{(1)}(z)$ and $\phi_k^{(2)}(\tilde{z})$ be generic, unrelated differentials? If not, how are they constrained? (Hint: pay attention to all the residues)

iii) Compute the difference $d_{k,n+m} - d_{k,n} - d_{k,m}$. Give an interpretation of the difference in the light of the previous point.

3 A spectral curve

In this section we will consider the properties of a complex curve Σ defined by the equation

$$x^k + F_1(z)x^{k-1} + F_2(z)x^{k-2} + \cdots + F_k(z) = 0 \tag{2}$$

where $\phi_a(z) = F_a dz^a$ are degree a differentials with degree a poles at some common set of n points z_i. There is a natural meromorphic one-form on this curve, $\lambda = x dz$ with simple poles on the points above the z_i.

i) Check this statement, compute the residues of λ

The curve Σ depends on $\sum_{a=1}^{k} d_{a,n}$ parameters u_I. It is useful to consider the variation of λ as we vary the u_I.

ii) Prove that $\omega_I = \frac{\partial \lambda}{\partial u_I}$ is a holomorphic differential on Σ, with no poles.

iii) Compute the genus of Σ. Are the ω_I a basis for the holomorphic differentials?

The one form λ has interesting periods

$$Z_\gamma(u_I) = \oint_\gamma \lambda \tag{3}$$

where γ is some path in Σ. If γ is moved across a z_i, Z_γ will jump by the residue there.

4 Decoupling limit again

We are interested in the behavior of the periods as we apply the decoupling limit discussed in the previous problem: we start with $n+m$ points z_i and keep m within distance $|q|$ of $z=0$, require all the differentials ϕ_a to have good limits both in the z and $\tilde{z}$ coordinates as $q \to 0$.

i) What is the geometric relation between Σ and the two curves $\Sigma^{(a)}$ defined by the $q \to 0$, fixed z and $q \to 0$, fixed $\tilde{z}$ limits?

ii) The periods along a path γ which stays at $|z| > 1$ go in the limit to periods of cycles of $\Sigma^{(1)}$. A similar statement applies to periods along a path which stays in the region $|z| < q$. What is the behavior of periods along paths which extend across the region between $\Sigma^{(1)}$ and $\Sigma^{(2)}$? Estimate the leading q dependence as $q \to 0$.

5 Sources for the lectures

Besides the sources listed by G.Moore, it will be useful to review basic facts about Riemann surfaces, their moduli spaces and pair of pants decompositions. For example, chapters 9.2, 9.3 of Polchinski vol. 1, and exercise 9.4