INTEGRABILITY IN GAUGE AND STRING THEORY

LECTURES 2 & 3.

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- WHY $\mathcal{N}=4$ SYM?

- GAUGE THEORIES IN NATURE.

- $\mathcal{N}=4$ IS THE **SIMPLEST**

  ↑

  MOST SYMMETRIC : - SUPER SYMMETRY
  - CONFORMAL INVARIACE
  - S-DUALITY

  → INTEGRABILITY IN THE PLANAR LIMIT

- IT IS THE "HYDROGEN ATOM" (OR HARMONIC OSCILLATOR) OF GAUGE THEORIES.

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$\mathcal{N}=4$ SUPER YANG MILLS

(- HYPERBOLIC COW OF GAUGE THEORIES ☺)

- $\mathcal{N}=2$ VECTOR MULTIPLET + HYPERMULTIPLET IN THE ADJOINT.

- IN $N=1$ NOTATION.

$$\mathcal{L} = \frac{1}{g^2}\int d^2\theta \; Tr[W^2] \; + \frac{1}{g^2}\int Tr[\phi h \tilde{h} - h \phi \tilde{h}]$$

~~$\mathcal{N}=2$ SUSY RELATES THIS COEFFICIENTS~~

. HAS SU(3) GLOBAL SYMMETRY (FROM $\mathcal{N}=1$ POINT OF VIEW

~~DOES NO~~

$$\phi = \phi_1 \qquad h = \phi_2 \qquad \tilde{h} = \phi_3 \qquad W = \varepsilon_{ijk} Tr[\phi^i \phi^j \phi^k]$$

. THIS SO(3) DOES NOT COMMUTE WITH $\mathcal{N}=2$ SUSY.

$\Rightarrow \mathcal{N}=4$ SUSY.

. FIELD CONTENT. - $SU(4) \approx SO(6)$ GLOBAL R-SYMMETRY

A_μ

ϕ^I SIX SCALARS. $I=1,\dots,6$

$\psi_{\alpha i}$ 4 - WEYL FERMIONS ($\&$ $\bar{\psi}_{\dot{\alpha}}^{\;i}$ OF COURSE)

$i=1,\dots,4$

- REPRESENTATIONS.

- OPERATORS @ ORIGIN $\longleftrightarrow$ STATES ON CYLINDER.

$\Delta \longleftrightarrow E.$

- P_μ RAISES E. ; K_μ LOWERS E.

- FIND LOWEST E STATE BY APPLYING K_μ.

$$K_\mu \underbrace{|\text{LOWEST}\rangle}_{\text{PRIMARY.}} = 0 \quad \rightarrow \quad \mathcal{O}(0)$$

~~APPLY~~ $P_\mu |\text{LOWEST}\rangle \rightarrow \partial_\mu \mathcal{O}(0)$ DESCENDENTS.

$P_\mu P_\nu |\text{LOWEST}\rangle \quad \partial_\mu \partial_\nu \mathcal{O}(0).$

$\vdots$

- CALCUTATE NORMS. $(P_\mu |\text{STATE}\rangle)^\dagger \rightarrow \langle\text{STATE}| K_\mu$

$\rightarrow$ ALL NORMS SHOULD BE POSITIVE

$\rightarrow$ CONDITIONS. IF $S=0 \rightarrow \Delta \geq 1$ & $\Delta = 1$ IS A FREE FIELD.

$S>0 \quad \Delta - S \geq 2$ & $\Delta - S = 2$ IS A CONSERVED CURRENT

ex: $S=1 \rightarrow \Delta = 3$
$S=2 \rightarrow \Delta = 4$
$\downarrow$
STRESS TENSOR

3

SUPER SYMMETRY.

$Q_{\alpha I}$, $Q_{\dot\alpha}{}^{I}$ I 1,..4 $\to$ 16 SUPERCHARGES.

$[K_\mu , Q_{\alpha I}] \sim S_{\dot\alpha I}$ NEW SPECIAL CONFORMAL SUSY.

$\Rightarrow$ 16 MORE SUPER SYMMETRIES.

$$\{ Q , S \} \approx D + \underset{\text{LORENTZ}}{M} + \underset{SO(6)}{\underbrace{J}} \text{ CHARGES.}$$

$Q \to D = 1/2$
$S \to D = -1/2$ $\{S,S\} \approx K$ AS $\{QQ\} \approx P$

- FORM A SUPERGROUP.

  $SO(6) \sim SU(4)$
  $SO(2,4) \sim SU(2,2)$

  SUPER $\to$ $SU(2,2|4)$ $\rightsquigarrow$ $PSU(2,2|4)$

  $\sim \left(\begin{array}{c|c} 1 & \\ \hline & 1 \end{array}\right)$

- SIZE OF REPRESENTATION:

  - $D \gg$ SPIN, R-CHARGE $\to$ Q & S LIKE CREATION & ANNIHILATION OPERATORS & 16 OF EACH.

    2^{16} STATES.

  - RANGE OF SPINS. Q_α $Q_{\pm 1/2, i}$ $\to$ 4 OF THEM. -2 TO 2 in $SU(2)_L$
    $Q_{\dot\alpha}$ $Q_{\pm 1/2}{}^{i}$ " -2 TO 2 in $SU(2)_R$.
    $S = J^3_L + J^3_R \to -4, 4$

  - ALL LONG REPRESENTATIONS CONTAIN HIGH SPIN STATES.

    IN AdS $\to$ SHOULD BE MASSIVE STRING STATES

- ALL SUPERGRAVITY MODES IN ADS. ARE IN SHORT REPRESENTATIONS.

  - STRESS TENSOR MULTIPLET $\subset$ SUPERMULTIPLET

    $\rightarrow \mathrm{Tr}[\phi^{(I}\phi^{J)}]$ IS HIGHEST WEIGHT (LOWEST ENERGY).

    HAS $\Delta = J = 2$

    $\uparrow$ SPIN IN SO(6).

  - $\mathrm{Tr}[\phi^{(I_1} \cdots \phi^{I_J)}]$ $\quad \Delta = J$

  - THESE ARE ½ BPS REPRESENTATIONS.

    $2^{\frac{16}{2}} \sim 2^8 = 256$ STATES FROM Q's

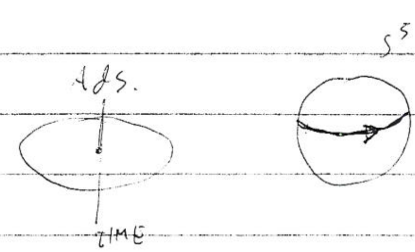
- ~~ALL SUGRA MODES CORRESPOND TO STATES IN~~ THE ABOVE MULTIPLETS. $\rightarrow$ ALL KK MODES ON S^5

- ARE THESE <u>ALL</u> THE PROTECTED (SINGLE TRACE) STATES

  YES ✓

- $\phi^5 + i\phi^6 = Z$

- $\mathrm{Tr}[Z^J]$ IS A BPS STATE.



- ARE THERE NON-PROTECTED STATES?

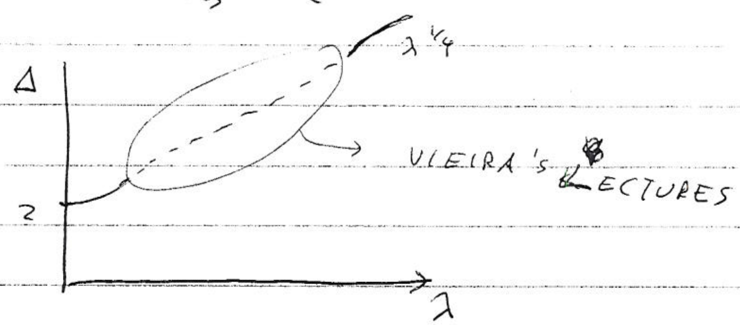
  → MANY!

$\Delta = 2$ $\quad$ $\mathrm{Tr}\left[\phi^I \phi^I\right]$ = "KONISHI"

$S=0$, $J=0$

$$\Delta = 2 + \# g^2 N + \cdots \qquad g^2 N \ll 1.$$

$$\Delta \sim \frac{R}{\ell_s} \sim \left(g^2 N\right)^{1/4} \qquad g^2 N \gg 1.$$

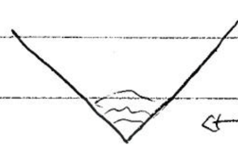


- HIGH SPIN.

$$\mathrm{Tr}\left[\phi \overleftrightarrow{\partial}^S \phi\right] \qquad \Delta = 2 + S + \# g^2 N.$$

$$S \to \infty \qquad \Delta - S \simeq 2 + \Gamma_{cusp}(\lambda) \log S$$

Γ_{cusp} ALSO APPEARS IN AMPLITUDES & WILSON LOOPS.



$\leftarrow$ DIV. $\simeq e^{-\Gamma_{cusp}\left[\log(\text{CUT OFF})\right]^2}$

$$\Gamma_{cusp} \sim \sqrt{\lambda} \qquad \lambda \gg 1.$$

- CAN WE COMPUTE THE ANOMALOUS DIMENSIONS FOR ALL LOOPS?

  - FEYMAN DIAGRAMS → HARD! (EVEN PLANAR ONES)

  - PLANAR LIMIT → THEORY HAS EXTRA SYMMETRY

  "INTEGRABILITY"

  - SEEN @ 1 LOOP & @ STRONG COUPLING.

- EXHIBIT ONE MORE SYMMETRY.

$$\bullet\ \frac{dx^2+dz^2}{z^2} \xrightarrow[\text{T-DUALITY IN } X]{} z^2dy^2 + \frac{dz^2}{z^2} \xrightarrow[z=1/r]{} \frac{dy^2+dr^2}{y^2}$$

$SO(2,4)$ (LEGAL IN PLANAR THEORY) $\widetilde{SO}(2,4)$.

NON-LOCAL SYMMETRY
||
EXTRA SYMMETRY.

SPECIAL CONFORMAL

THIS CAN BE GENERALIZED TO THE FULL $AdS_5 \times S_5$ σ-MODEL.
& BE DONE AT THE PATH INTEGRAL LEVEL.

- HOW DO WE APPLY IT TO EXTRACT ANSWERS?

- INTEGRABILITY: IN CLASSICAL MECHANICS
  ↑
  SAME NUMBER OF CONSERVED CHARGES AS DEGREES OF FREEDOM.

- IN QUANTUM THEORIES → SET OF COMMUTING OPERATORS.

- IN QUANTUM FIELD THEORIES.
  - ∞ NUMBER OF CONSERVED CHARGES (HIGHER SPIN)
  - IN 1+1 DIMENSIONS (DUE TO COLEMAN-MANDULA)

- PLANAR GAUGE THEORY HAS A 1+1 DIM. STRUCTURE
  ↕
  STRING IS 1+1 DIM.

- IT IS INTEGRABLE FOR $\mathcal{N}=4$ SYM & $AdS_5 \times S_5$.
  (PROBABLY ALSO FOR $AdS_4 \times \mathbb{CP}^3$ ...).

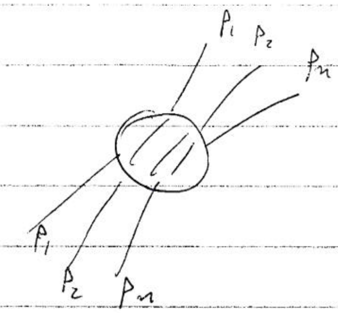
⇒ WE CAN COMPUTE SOME QUANTITIES EXACTLY.

~~INTEGRABILITY~~

<u>GOAL</u>:
- SOME IDEAS
- SOME TRICKS
- THE REALLY SIMPLE EXPLANATIONS ARE YET TO BE FOUND!
- MUCH-MUCH-MUCH SIMPLER THAN ANY OTHER METHOD!

INTEGRABILITY IN 1+1 THEORIES (RELATIVISTIC, MASSIVE THEORIES).

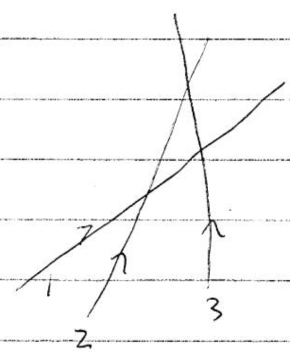
- COLEMAN MANDULA
  - $\Rightarrow$ TRIVIAL S-MATRIX $d>2$.
  - $\nexists$ IN $d=2$.
  - $\Rightarrow$ FACTORIZED SCATTERING.



$\{p_i\}_{in} \to \{p_i\}_{out}$ SAME SET.

- NO PARTICLE CREATION

- LABELS OF INTERNAL QUANTUM NUMBERS CAN CHANGE (PARTICLES OF SAME MASS)

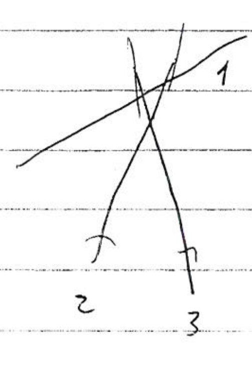


$|123\rangle \rightarrow |123\rangle$

$$S_{12}\,S_{13}\,S_{23} = S_{23}\,S_{13}\,S_{12}$$

YANG BAXTER EQN.

(NONTRIVIAL ONLY IF S IS A MATRIX).



- WHOLE THEORY IS SOLVED BY GIVING THE 2-PARTICLE S-MATRIX -

- METHODS:

1. START FROM AN INTEGRABLE LATTICE MODEL & DERIVE THE CONTINUUM LIMIT

2. BOOTSTRAP

    - DERIVE OR GUESS THE SPECTRUM OF MASSIVE PARTICLES

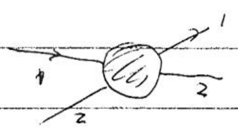
    - WRITE CROSSING EQUATION FOR THE S-MATRIX

    - ANALYTICITY

    - GUESS S-MATRIX

    - CHECK ANALYTIC PROPERTIES
      → ALL POLES (IN PHYICAL REGION) SHOULD HAVE AN INTERPRET.

- CROSSING.



$$A(12 \to 12) \xrightarrow[p_1^\mu \to -p_1^\mu]{} A(\bar{1}2 \to \bar{1}2)$$

- NOTATION. $p^\mu = (m\cosh\theta, m\sinh\theta)$ $\quad \theta$ = RAPIDITY.

$S(\theta_1 - \theta_2)$ BY LORENTZ SYMMETRY -

- EXAMPLE.

  - SINH- GORDON $\qquad S = \frac{1}{g^2}\int (\partial\phi)^2 + \mu \, ch(\phi).$

  - 1-PARTICLE ~~OF MASS~~ m.

  - S-MATRIX $\rightarrow$ PHASE $\quad S(\theta)$.

  - UNITARITY $\qquad S(\theta)\, S(-\theta) = 1$

    (UNITARITY $S(\theta)^* = \frac{1}{S(\theta)}$ + REAL ANALYTICITY $S(-\theta) = S(\theta)^*$)

  - CROSSING $\qquad S(\theta)\, S(\theta - i\pi) = 1$

SIMPLEST $\quad S = \pm 1 \rightarrow$ TOO SIMPLE.

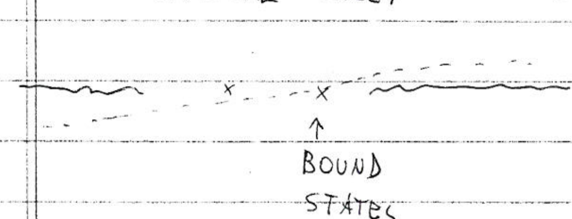
~~NEXT SIMPLEST~~ ~~1 POLE.~~

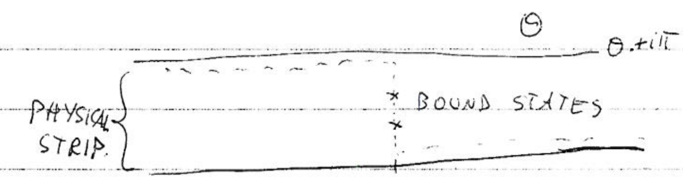
$$S = \frac{sh\,\theta - i \sin \pi p}{sh\,\theta + i \sin \pi p} \qquad p \sim \frac{g^2}{1+g^2}$$

(see hep-th/0005181 -)

$$s = (p_1 + p_2)^2 = m^2\left[2 + 2\, ch\,\theta\right]$$

PHYSICAL SHEET $\quad s$.





THIS S MATRIX HAS NO POLES IN THE PHYSICAL STRIP.

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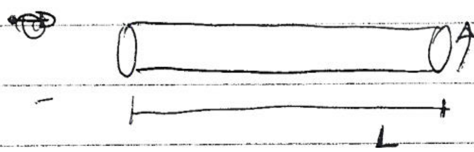
- THIS SOLVED THEORY ON AN ∞ LINE.

- THEORY ON A CIRCLE ?

- CASIMIR ENERGY
- SPECTRUM OF STATES.

β



β

$$Z = e^{-L \boxed{\beta f(\beta)}}$$

$E(\beta)$

CASIMIR ENERGY

- THERMODYNAMICS $\rightarrow$ - FIND STATES ON LARGE CIRCLE.
  & SUM.

- LARGE CIRCLE

L

BETHE EQUATIONS.

$$\boxed{e^{i p_j L} \prod_{k \neq j} S_{jk} = 1}$$

$j = 1 \ldots$, NUMBER OF PARTICLES.

NEED SOME KEY IDEAS.

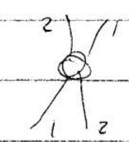
1) DECOMPACTIFY THE WORLDSHEET 

$J \to \infty$

2) FIND THE ELEMENTARY EXCITATIONS AROUND A SIMPLE "VACUUM".

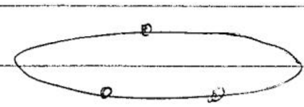
3) FIND THEIR DISPERSION RELATION $\mathcal{E}(p)$.

4) FIND THEIR 2-PARTICLE S-MATRIX

$S(p_1, p_2)$ 

- ANALYTICITY
- CROSSING

5) COMPACTIFY AGAIN ON A VERY LONG CIRCLE



6) GO TO A SMALL CIRCLE.

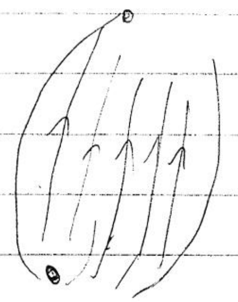
12.5

SUMMARY.

1 COMPUTE ANOMALOUS DIMENSIONS.

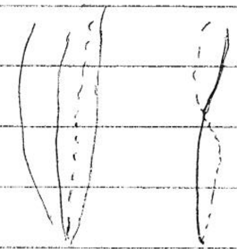
STEPS: - DECOMPACTIFY & PRODUCE A

$J \to \infty$ "MASSIVE" σ-MODEL



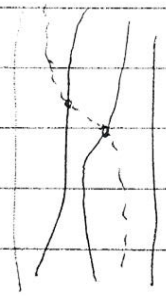
$\mathrm{Tr}[Z^J] \,..\, \mathrm{Tr}[\bar{Z}^J]$

- PRODUCES A WEB, A SCAFFOLDING, WHERE OTHER PARTICLES WILL MOVE
- PRODUCES SITES ON A CHAIN.



$Z \,..\, WZ \,...$

- PLANARITY: - MOVES LOCALLY
  - LOCALITY ON THE CHAIN.



- MOVES FURTHER @ HIGHER LOOPS -

FIND:
- DISPERSION RELATION

&

- S-MATRIX

(QCD (PURE YM) ANALOGY → TAKE A STRING & STRETCH IT)

$\boxed{13}$

- LARGE J LIMIT.

- $\mathrm{Tr}[Z^J]$ $\quad J \to \infty$

- BPS. $PSU(2,2|4) \to SU(2|2)^2$

BOSONIC.

$$\begin{array}{ccc} SO(4)_{AdS} & \times & SO(4)_{sph.} \\ \downarrow & & \downarrow \\ SU(2) \times SU(2) & & SU(2) \times SU(2) \end{array}$$

SU(2|2)

- KEEP ALL GENERATORS WITH $\Delta - J = 0$ $\quad \mathcal{E} = \Delta - J$

$\mathcal{E}$ = ENERGY ABOVE GROUND STATE.

- REMAINING GENERATORS HAVE $\mathcal{E} = \Delta - J = \pm 1$ $\rightarrow$ CREATE ZERO "MOMENTUM" MODES

$$J_z^i \quad \mathrm{Tr}[Z^J] \rightarrow \mathrm{Tr}[W Z^{J-1}]$$

$$J\,J \quad \rightarrow \sum_{\ell_0} \mathrm{Tr}[W Z^{\ell} W Z^{J-2-\ell}]$$

- NON ZERO MOMENTUM.

$$\sum_{\ell} \mathrm{Tr}[W Z^{\ell} W Z^{J-2-\ell}]\, e^{ip\ell}.$$

$$\sum_{\ell} Z \cdots Z \underset{\uparrow_{\ell}}{W} Z \cdots Z \quad e^{ip\ell}$$

- TOTAL MOMENTUM $= 0$ $\to$ CYCLICITY OF TRACE.

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- ELEMENTARY EXCITATIONS.

ACT WITH J^i 4 $Z \cdots \phi^i Z \cdots$

ACT WITH P_μ 4 $Z \cdots \partial_\mu Z\, Z \cdots$

ACT WITH BROKEN SUSY   8 $Z \cdots \psi_{\frac{1}{2}} Z \cdots Z$

ψ w/ CHARGE $J = \frac{1}{2}$.

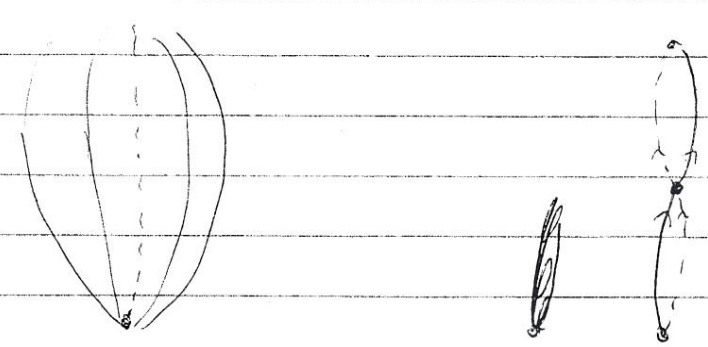
$\Delta - J = \varepsilon = 1$ FOR THESE

(OTHERS $Z \cdots \bar{Z} Z \cdots Z$ $\rightarrow$ DECAY INTO THE ABOVE ONE.

$Z \cdots \partial^\mu Z \cdots Z \cdots$

- DISPERSION RELATION

$$\sum_{\ell} Z Z \underset{\uparrow_\ell}{W} Z \cdots Z \; e^{ip\ell} \quad = \mathcal{O}.$$



$\bar{Z}\bar{W}ZW$

DIAGRAM FOR ORDER CHANGE $\sim \mathrm{Tr}\left([ZW][\bar{Z}\bar{W}]\right)$
(FROM F-TERMS OR SUPERPOTENTIAL)

$$g^2 N \times \int d^4x \frac{1}{|x-x_1|^4} \frac{1}{|x-x_2|^4} \underset{\substack{\text{DIV} \\ \text{NEAR} \\ x_1 \sim x_1}}{\sim} g^2 N \log\left(\frac{x}{\Lambda}\right) \times \frac{1}{(x_1-x_2)^4} \cdots$$

- $\Delta - J = 1 + \lambda\left[e^{iP} + e^{-iP} - 2\right] \sim 1 + \#\lambda \sin^2 P/2 + \lambda^2 + \cdots$

- PERIODIC IN P $p \sim p + 2\pi$ (LIKE A LATTICE) → Z's PROVIDE A BACKGROU
- THE PARTICLES SHOULD TRANSFORM UNDER THE UNBROKEN SYMMETRY $SU(2|2)^2$.

$SU(2|2)$ $Q_{\alpha\dot\beta}$ → 4 ← COMPLEX
$S^{\alpha\dot\beta}$

dim OF NON BPS $\sim 2^4 = 16$

- IN OUR CASE → DIM 4 REPR.

$SU(2)_{SPHERE} \times SU(2)$

$(2, 1) = \phi_\alpha$ → BOSONS.
$+$
$(1, 2) = \psi_{\dot\beta}$ → FERMIONS.

- GOLDSTONES → 4×4.

$\underline{p \neq 0}$ → SAME NUMBER, $\varepsilon > 1$

- ⇒ Extra central CHARGES.
IF THEORY WAS RELATIVISTIC.

$Q\,Q = \Gamma E + \Gamma p$

$D=1$ $\{Q, Q\} = K$ $D=-1$ $\{S, S\} = \bar{K}$

~~$D=-1$ $\{Q, Q\} = \bar{K}$~~

$Q\,W \to \Psi$
$Q\,\Psi \to [W, Z]$ $K(\cdot) \sim [W, Z]$

$Z \ldots W Z \ldots \quad e^{ipl}$

$\quad\quad\quad\;\uparrow$
$\quad\quad\quad\; l$

QQ

$e^{ipl} Z \quad \underset{l}{\underset{\uparrow}{W}} Z \ldots \quad - \; Z \ldots \underset{l+1}{Z \underset{\uparrow}{W} Z} \ldots$

$e^{ipl} (1-e^{-ip}) \; Z \ldots \underset{l}{\underset{\uparrow}{W}} \; \underset{\uparrow}{Z} \; Z \ldots$

EXTRA Z to THE RIGHT.

$K(p) = \sqrt{gN} \; (1 - e^{-ip})$

. THE ALGEBRA HAS:

$\varepsilon^2 = 1 + K\bar{K}$ _ LOOKS RELATIVISTIC IN TERMS OF K.

$$\boxed{\varepsilon^2 = 1 + \frac{\lambda}{\pi^2} \sin^2 p/2}$$

_ IS NOT IN TERMS OF p, $\lambda = g_{YM}^2 N$

. IN PRINCIPLE $\lambda \to f(\lambda)$ $\to$ SO FAR NO NEED.

. p DEPENDENCE IS EXACT.

- PHASE OF K DEPENDS ON OTHER MOMENTA.

- SUMMARY SO FAR:
  - IDENTIFIED FUNDAMENTAL EXCITATIONS.

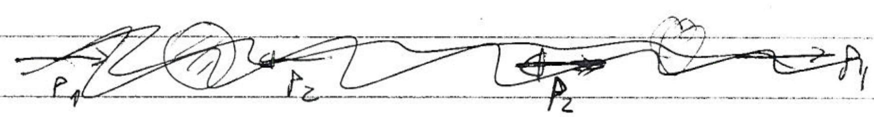
    → GOLDSTONE BOSONS FOR ~~PSU(2~~ $PSU(2,2|4) \to SU(2|2)^2$.

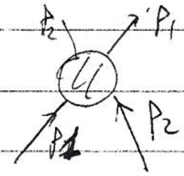
  - FOUND THE EXACT DISPERSION RELATION.

  - PARTICLES ARE IN AN IRREDUCIBLE SHORT REPRESENTATION OF $SU(2|2)^2$ (BPS)

    $$(\boxslash, \boxslash).$$

- 2 PARTICLE S-MATRIX

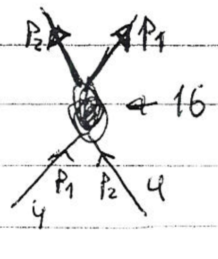




- DO ONE SU(2|2) FACTOR AT A TIME

  $$S = S_{1st} \otimes S_{2nd}.$$

- 4×4 STATES → $16 = 2^4$ = DIM OF NON-BPS REPRES.



S-MATRIX → UNIQUE MATRIX STRUCTURE

OVERALL PHASE → NOT YET FIXED

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- SAME ARGUMENT SAYS THAT 4pt AMPLITUDES OF MASSLESS PARTICLES IN $\mathcal{N}=4$ SYM OR $\mathcal{N}=8$ SUGRA ARE UNIQUELY FIXED UP TO AN OVERALL FUNCTION.

$$S = \underbrace{S_0(p_1, p_2, \lambda)}_{\text{NEED TO FIX THIS FACTOR}} \times \hat{S} \otimes \hat{S}$$

KNOWN MATRICES

- FIXING S_0:
  - PERIODIC FUNCTIONS OF p_i
  - $S_0(1,2) = \left(S_0(2,1)\right)^{-1}$ (FROM UNITARITY).
  - ANALYTIC STRUCTURE?

$$\varepsilon^2 = 1 + \lambda \sin^2 p/2 \quad \rightarrow \quad \varepsilon = \pm\sqrt{1 + \cdots}$$

2 SIGNS → LIKE IN RELATIVISTIC THEORIES.

⇒ CROSSING SYMMETRY.

$$\mathcal{A}(++ \longrightarrow ++)\Big|_{\substack{p_1 \to -p_1 \\ \varepsilon_1 \to -\varepsilon_1 \\ p_2, \varepsilon_2}} = \mathcal{A}(-+ \longrightarrow -+)\Big|_{\substack{p_1, \varepsilon_1 \\ p_2 \varepsilon_2}}$$

$$\underset{\text{CROSSED POINT}}{} S_0(1,2)\, S_0(\bar{1},2) = \text{KNOWN FUNCTION OF } P_1, P_2, \lambda -$$

FROM $\hat{S}$ → RATIO OF $\dfrac{A(++\to++)}{A(-+\to-+)}$.

- COMPUTE S_0 AT WEAK & STRONG COUPLING
  ↑ STRINGS IN $AdS_5 \times S_5$

- GUESS S_0

  S_0 = EXPLICIT FUNCTION OF P_1, P_2, λ.
  (SIMPLE INTEGRAL OF A Γ FUNCTION).

- PERFORM CONSISTENCY CHECKS ON SOME OF THE SINGULARITIES OF S_0

→ THE ∞ $\underset{\text{(LENGTH)}}{J}$ PROBLEM IS COMPLETELY SOLVED !

- S MATRIX IS ASYMPTOTIC → WAIT TILL PARTICLES ARE WELL SEPARATED.

(20) FINITE BUT LARGE J :

$J = L =$ LENGTH OF THE CHAIN.

- PARTICLES WITH A SCATTERING PHASE $S(p_1, p_2)$.

$$\Psi(x_1, x_2) = \begin{cases} e^{ip_1 x_1 + p_2 x_2} & x_1 < x_2 \\ S(1,2)\, e^{ip_1 x_1 + p_2 x_2} & x_1 > x_2 \end{cases}$$

. PERIODICITY $\Psi(x_1 + J, x_2) = \Psi(x_1, x_2)$.

$$\Rightarrow \quad S(1,2)\, e^{ip_1 L} = 1$$

MANY PARTICLES.

$$e^{ip_i L} \prod_{\substack{j=1 \\ j \neq i}}^{m} S(p_i, p_j) = 1$$ BETHE EQUATIONS. (BETHE ANSATZ.)

m EQUATIONS FOR m MOMENTA.

THERE IS A ~~UNIQUE~~ DISCRETE SET OF SOLUTIONS. ~~UP TO ... SCALE~~

FREE : $e^{ip_i L} = 1 \rightarrow p_i = \frac{2\pi m}{L}$ DISCRETE SET LABELED BY m.

$\rightarrow$ SIMILAR HERE.

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IN THE FULL PROBLEM.

- $\hat{S}$ IS A MATRIX $\longrightarrow$ NEEDS TO BE DIAGONALIZED

USE INTEGRABILITY AGAIN.

- SIMPLER CASES:

  - SUBSECTORS -

$Z \cdots W Z \cdot W Z$ $\qquad$ $W \to U(1) \subset SU(2)$

S: $W W \longrightarrow W W$ BY CHARGE CONSERVATION

& S IS A PHASE

- WE CAN RESTRICT TO THESE SUBSECTORS.

  - THE ABOVE ONE IS CALLED "SU(2)"

    - AT WEAK COUPLING $\to$ XXX SPIN CHAIN.

  - REDUCTION TO THE SUBSECTOR IS EXACT FOR LARGE J.

- SL(2) SUBSECTOR.

$Z \cdots \partial_+ Z Z \cdots Z \partial_+ Z \qquad Z \cdots Z \partial_+^2 Z \cdots$

- ALOWS US TO CONSIDER HIGH SPIN OPERATORS.

$\mathrm{Tr}[Z \overleftrightarrow{\partial}_+^S Z]$

$\mathrm{Tr}[Z^J \overleftrightarrow{\partial}_+^S Z]$

(22)

- $Z \,\partial_+ \cdots \partial_+\, Z$
- WE WILL DISTRIBUTE MANY DERIVATIVES

$$e^{ip_iL}\prod_{\substack{j=1\\ j\neq i}}^{n} S(p_i,p_j) = 1$$

$$p_i L + \sum_{j}\frac{1}{i}\log S(p_i,p_j) = 2\pi m_i$$

- LOWEST ENERGY STATES m_i INCREASE LIKE i

$m_i \sim i$

→ LIKE FERMIONS FILLING A FERMI SEA.

→ LARGE L → CONTINUUM APPROXIMATION.

$$\cancel{\frac{dp}{dm}L + \int dm'\frac{1}{i}\log S(p(m),p(m')) = 1}$$

$$L + \frac{1}{i}\int dm'\,\partial_{p_i}\log(p,p') = 2\pi \underbrace{\boxed{\frac{dm}{dp}}}_{\rho(p)}$$

$$L + \int dp'\; K(p,p')\,\rho(p') = 2\pi\rho(p)$$

$K(p,p') = \frac{1}{i}\partial_p \log S(p,p')$

- INTEGRAL EQUATION FOR $\rho(p)$.
- COMPUTE ENERGIES:

$$\mathcal{E} = \sum_i \mathcal{E}(p_i) = \int dm\, \mathcal{E}(p(m)) = \int \rho(p)\,\mathcal{E}(p)\,dp$$

- THE PROBLEM IS JUST SOLVING THE INTEGRAL EQUATION.