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Some Suggested Preliminary Reading and Problems

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Abstract: Here are some problems to think about together with a list of some papers which will serve as background and source for the PiTP lectures of G. Moore. They will also be relevant to the lectures of D. Gaiotto.

Contents

1. Problems

1.1 Angular momentum of a pair of dyons

Consider two dyons of (magnetic, electric) charge (p_i, q_i), $i = 1, 2$.

a.) By computing the Poynting vector of the electromagnetic field show that the two-dyon system carries classical angular momentum (in cgs units) (♣ around what origin? ♣)

$$\vec{J} = \frac{1}{c}(p_1 q_2 - p_2 q_1)\hat{r} \tag{1.1}$$

where $\hat{r}$ is the unit vector pointing from dyon 2 to dyon 1.

b.) Using quantum mechanical quantization of angular momentum conclude that

$$(p_1 q_2 - p_2 q_1) = \frac{\hbar c}{2} n \tag{1.2}$$

where n is an integer.

c.) Show that the antisymmetric bilinear form

$$\langle (p_1, q_1), (p_2, q_2) \rangle := p_1 q_2 - p_2 q_1 \tag{1.3}$$

defines a *symplectic form* on $\mathbb{R}^2$.

1.2 Group Theory

a.) Let ρ_n denote the n-dimensional representation of $SU(2)$. What is the maximal spin?

b.) The character of a representation V of $SU(2)$ is defined to be $\chi_V(y) = \mathrm{Tr} y^{2J_3}$ where J_3 is any generator. Show that

$$\chi_{\rho_n}(y) = \frac{y^n - y^{-n}}{y - y^{-1}} \tag{1.4}$$

Evaluate the limits $y \to \pm 1$ using L'Hopital's rule.

c.) An arbitrary finite dimensional representation of $SU(2)$ is completely reducible and hence isomorphic to $\sum_{n\geq 1} a_n \rho_n$ for some integers $a_n \in \mathbb{Z}_+$. Show that the character of a representation V of $SU(2)$ determines V uniquely up to isomorphism.

d.) A *virtual* representation is a formal sum $\sum a_n \rho_n$ where a_n are integers. Show that the virtual representations form a ring. Show that the character of a virtual representation does not determine it uniquely.

1.3 $N = 2$ Algebra and its BPS Bound

We follow the conventions of Bagger and Wess [REF] for $d = 4, \mathcal{N} = 1$ supersymmetry. In particular $SU(2)$ indices are raised/lowered with $\epsilon^{12} = \epsilon_{21} = 1$. Components of tensors in the irreducible spin representations of $so(1,3)$ are denoted by $\alpha, \dot\alpha$ running over $1,2$. The rules for conjugation are that $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger$ and $(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha}$.

The $\mathcal{N} = 2$ supersymmetry operators are $(Q_\alpha{}^A, \bar Q_{\dot\alpha B})$ where A, B are $SU(2)_R$ indices running from 1 to 2. They satisfy the Hermiticity conditions

$$(Q_\alpha{}^A)^\dagger = \bar Q_{\dot\alpha A} \tag{1.5}$$

and the $\mathcal{N} = 2$ algebra

$$\begin{aligned} \{Q_\alpha{}^A, \bar Q_{\dot\beta B}\} &= 2\sigma^m_{\alpha\dot\beta} P_m \delta^A_{\ B} \\ \{Q_\alpha{}^A, Q_\beta{}^B\} &= 2\epsilon_{\alpha\beta}\epsilon^{AB} \bar Z \\ \{\bar Q_{\dot\alpha A}, \bar Q_{\dot\beta B}\} &= -2\epsilon_{\dot\alpha\dot\beta}\epsilon_{AB} Z \end{aligned} \tag{1.6}$$

where Z is the central charge and P_m is the Hermitian energy-momentum vector with $P^0 \geq 0$.

a.) Check that the above commutation relations are consistent with Hermiticity.

b.) Define

$$\mathcal{R}_\alpha{}^A = \xi^{-1} Q_\alpha{}^A + \xi \sigma^0_{\alpha\dot\beta} \bar Q^{\dot\beta A} \tag{1.7}$$

Here ξ is a phase: $|\xi| = 1$. Show that these operators satisfy the Hermiticity conditions

$$\begin{aligned} (\mathcal{R}_1{}^1)^\dagger &= -\mathcal{R}_2{}^2 \\ (\mathcal{R}_1{}^2)^\dagger &= \mathcal{R}_2{}^1 \end{aligned} \tag{1.8}$$

and the algebra

$$\{\mathcal{R}_\alpha{}^A, \mathcal{R}_\beta{}^B\} = 4\left(E + \mathrm{Re}(Z/\zeta)\right)\epsilon_{\alpha\beta}\epsilon^{AB} \tag{1.9}$$

where $\zeta = \xi^{-2}$.

c.) By choosing a suitable phase ξ deduce the BPS bound $E \geq |Z|$.

d.) Introduce $T^A_\alpha = \xi^{-1} Q^A_\alpha - \xi \sigma^0_{\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}A}$. For your special value of ξ interpret the T and $\mathcal{R}$ supersymmetries as those which are "preserved" and "broken" on quantum states saturating the BPS bound.

1.4 BPS representations

♣ Problem on BPS reps and their characters. ♣

1.5 Reduction of a $U(1)$ gauge field to three dimensions and dualization

Consider a $U(1)^r$ gauge field on $\mathbb{R}^{1,2} \times S^1$ with metric $ds^2 = dx^\mu dx_\mu + R^2 (dx^3)^2$ and $x^3 \sim x^3 + 2\pi$. The action is

$$\int -\frac{1}{4\pi} \mathrm{Im}\tau_{IJ} F^I * F^J + \frac{1}{4\pi} \mathrm{Re}\tau_{IJ} F^I F^J \tag{1.10}$$

where $I, J = 1, \dots r$, F^I is the 2-form fieldstrength and τ_{IJ} is a symmetric complex matrix with positive definite imaginary part. It may be spacetime-dependent.

Show that the low energy effective action in three dimensions is a sigma model with a torus as target space and action

$$\int -\frac{1}{2R} (\mathrm{Im}\tau)^{-1,IJ} dz_I * d\bar{z}_J \tag{1.11}$$

where $dz_I = d\varphi_{m,I} - \tau_{IJ} d\varphi_e^J$ where φ_e^I and $\varphi_{m,J}$ are real scalar fields with period 1.

Hints:

Consider the dimensional reduction to $\mathbb{R}^3$. Write $F^I = d\varphi_e^I \wedge dx^3 + \bar{F}^I$ where φ_e^I is a scalar in $\mathbb{R}^{1,2}$ with period 1. Dualize the 3d vector field with fieldstrength $\bar{F}^I$ by introducing

$$\exp i \int \bar{F}^I d\varphi_{m,I} \tag{1.12}$$

into the path integral and integrating out $\bar{F}^I$ through a Gaussian integral.

For more help see [14] and [13].

1.6 Dual torus

Let Γ be a symplectic lattice of rank r.

a.) Show that $T := \Gamma^* \otimes \mathbb{R}/\mathbb{Z}$ is an algebraic torus of dimension r, i.e. it is isomorphic to $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (with r factors).

b.) Show that for any vector $\gamma \in \Gamma$ there is a canonical $\mathbb{C}^*$-valued function X_γ on T.

c.) Show that T has a holomorphic symplectic form, and express it in terms of functions X_γ.

1.7 Reduction of Hitchin equations

Write the self-dual Yang-Mills equations in four-dimensions and dimensionally reduce to 2 dimensions along two dimensions transverse to the lightcone.

Show that these equations take the form

$$\begin{aligned} F_{z\bar{z}} + [\Phi_z, \bar{\Phi}_{\bar{z}}] &= 0 \\ \partial_{\bar{z}}\Phi_z + [A_{\bar{z}}, \Phi_z] &= 0 \end{aligned} \tag{1.13}$$

where $z, \bar{z}$ are lightcone coordinates.

WARNING: I haven't set conventions carefully here yet.

2. Some Sources for the Lecture

The course will cover material primarily from papers by Denef and Moore and by Gaiotto, Moore, and Neitzke.

A previous knowledge of some aspects of N=2 susy and of the attractor mechanism and the split attractor flows would be helpful.

For general background on N=2 supergravity, special geometry, the attractor mechanism, and black hole entropy see [8].

The viewpoint on the attractor mechanism we will use is reviewed in Section 2 of [9].

For a nice introductory discussion of split attractor flows see [1].

In lecture one we will begin with wall-crossing formulae from the viewpoint of supergravity. For a brief qualitative overview see [3]. More details are in [2].

For essential background for the paper [4] see

Nigel Hitchin, "Hyperkahler manifolds," Seminaire N. Bourbaki, 1991-1992, exp. no. 748, p.137-166.

available online at http://www.numdam.org for a nice review of hyperkahler geometry.

A key role will be also played by reduction of N=2 theory from four to three dimensions. We recommend [10].

For the paper [6] an important role will be played by a hypothetical six-dimensional superconformal theory. For some background on this theory see [11, 16].

The essential geometrical construction of some N=2 d=4 theories from M5 branes was introduced by Witten in [15]. See [6], section 3 and [5] for further explanation and development.

The geometrical picture of BPS states in this context was first discussed in [7] and some nice aspects of wall-crossing in a special class of theories was discussed in [12]. This geometrical picture for BPS states is used extensively in [6].

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