Homework

Lectures by Edward Witten

(1) (a) Consider a four-dimensional theory with $\mathcal{N}=1$ supersymmetry and two free massless chiral superfields Φ_1, Φ_2. By analyzing the global symmetries (somewhat as done in the lecture for a combination of an $\mathcal{N}=1$ vector multiplet and an $\mathcal{N}=1$ chiral multiplet) show that this theory actually has $\mathcal{N}=2$ supersymmetry.

There is one thing that might make this exercise slightly tricky. If you literally write down all of the global symmetries of the model, there are too many to be useful. I suppose that without giving too much away, I can point out that the full global symmetry group of the model includes $SU(2)^3 \times U(1)$, but that there is an $SU(2)^2 \times U(1)$ subgroup that is most useful. Here one $SU(2)$ acts on scalars only, and the second on both scalars and fermions. What does the $U(1)$ act on?

(b) In addition to the $SU(2)^3 \times U(1)$, can you see some spontaneously broken global symmetries? Answer this question just for the theory of a single massless chiral superfield Φ. And (without changing the Lagrangian of the theory) can you see how to modify the usual supercurrent of this theory (for simplicity still for $\mathcal{N}=1$ with a single chiral superfield) so as to get a theory with spontaneously broken supersymmetry, the fermion becoming a Goldstone fermion? A Goldstone fermion is a fermion that appears as a pole in the two point function of the supercurrent. Usually, when there is a Goldstone fermion, bosons and fermions have different masses; is that so here?

(c) Returning to (a), a more advanced form of the exercise is to write the conserved supercurrent that generates $\mathcal{N}=2$ supersymmetry and show that it does have the right commutation relations. Let me suggest using an $SU(2) \times SU(2)$-invariant notation in doing this.

(d) Now consider the case of giving equal masses to Φ_1 and Φ_2 by adding a superpotential $W = m\Phi_1\Phi_2$, or equivalently, up to a redefinition of the fields, $W = m(\Phi_1^2 + \Phi_2^2)$. Does the model still have $\mathcal{N}=2$ supersymmetry? How much of the global symmetry is still present?

(2) This exercise is intended to fill a gap in the explanation in the lectures concerning the electric charge of a magnetic monopole.

Consider a theory with $SU(2)$ spontaneously broken to $U(1)$ by a scalar field ϕ in the adjoint representation. We assume that at infinity ϕ has an expectation value

$$\phi \to \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \text{ for } r \to \infty, \tag{1}$$

up to a gauge transformation. (We will write $\vec{x}$ for coordinates of $\mathbb{R}^3$ and $r = |\vec{x}|$.) However, in the field of a magnetic monopole, the gauge transformation required to put ϕ in the stated form depends on the direction in which we go to infinity. A better formula is

$$\phi \to \frac{f(r)}{r} \vec{\sigma} \cdot \vec{x}, \tag{2}$$

where $f(r)$ is a smooth function that is proportional to r for small r and approaches the limit a at infinity. (The $\vec{\sigma}$'s are the usual Pauli sigma matrices.) If one goes to infinity along the z axis, (2) is equivalent to (1); in general, if one goes to infinity in any direction, they are equivalent up to a gauge transformation. But there is no way to make a continuous gauge transformation that turns (2) into (1) everywhere; if you try to do that, you will generate the Dirac string singularity. The whole idea of the 't Hooft-Polyakov monopole in nonabelian gauge theory is that the Dirac string singularity goes away when $U(1)$ is embedded in a simple Lie group such as $SU(2)$, even though $SU(2)$ is broken to $U(1)$ at low energies.

The exercise we are going to carry out has nothing to do with supersymmetry. In a nonsupersymmetric theory, ϕ might be a hermitian field and then a would be real. Even if ϕ is a general complex field (not hermitian or antihermitian) as in $\mathcal{N} = 2$ supersymmetry, we can after redefining ϕ by a phase assume that a is real. In this case, only the hermitian part of ϕ is relevant in the following discussion, and for brevity I will just use the notation "ϕ" for this hermitian part.

Electric charge is the conserved quantity that is generated by a gauge transformation that preserves the value of the Higgs field ϕ at infinity. Let us write $\varepsilon = \sum_a \varepsilon^a T_a$ (here T_a are the generators of the Lie algebra of the gauge group) for the generator of such a gauge transformation. As usual, the infinitesimal gauge transformation generated by ε acts on a charged field Φ by $\delta\Phi = -i\varepsilon\Phi$ (where ε acts on Φ in the appropriate representation, and we need the $-i$ if we consider ε to be hermitian, as is conventional in the physics literature; actually many formulas of gauge theory look more natural if the Lie algebra is understood to consist of antihermitian matrices, in which case there is no $-i$ in the formula – but we will keep the $-i$ here). But what is a gauge transformation that preserves the Higgs field at infinity? The generator of such a gauge transformation is ϕ itself (since ϕ commutes with itself), except that we probably want to normalize the generator to remove the factor of a at infinity. So we take

$$\varepsilon = a^{-1}\phi. \tag{3}$$

Notice that we are defining electric charge so that a W boson has charge 2, and a "quark" (in the fundamental representation of $SU(2)$) has charge 1. (Warning: in the lecture, I use a normalization in which the electric charge of a W boson is 1 and that of a quark is 1/2. I think the convention used here makes the explanation of the topology more straightforward. Seiberg and I also, naturally enough, found the present convention to be useful when we considered the theory with elementary quark fields included; this case was discussed in our second paper on $\mathcal{N} = 2$ dynamics but there won't be time for it in the lectures.)

You might have been tempted to use

$$\widetilde{\varepsilon} = a^{-1}\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \tag{4}$$

where the matrix given is the vacuum expectation value of ϕ at infinity, so as to define $U(1)$ as an "unbroken global gauge transformation," i.e. a gauge transformation whose generator is a constant. This is probably the most elementary explanation given in textbooks of the unbroken gauge symmetry when $SU(2)$ is broken to $U(1)$: it is the gauge symmetry that leaves fixed the vacuum, i.e. the state in which ϕ is constant. The trouble with this approach is that in the field of a magnetic monopole, it is impossible to go to a gauge in which, even at infinity, ϕ is everywhere of the diagonal form that would commute with (4). So if we want to study the electric charge of a magnetic monopole, we cannot use this kind of definition. We use instead (3), since it makes sense for any finite energy configuration, including those with magnetic charge.

Now having arrived at the right definition of electric charge Q_e (as the gauge transformation generated by ε), we would like to ask if it is integer-valued. Equivalently, we want to know if the operator $\exp(2\pi i Q_e)$ is equal to 1. This operator is the gauge transformation

$$g(\vec{x}) = \exp(-2\pi i\varepsilon). \tag{5}$$

You should immediately note that $g(\vec{x})$ is not identically equal to 1.

(a) Show, however, that g approaches the limit 1 for $\vec{x} \to \infty$.

The importance of (a) is that we can "compactify" the region at infinity, so that space becomes S^3. Then we can think of g as a map from S^3 to the $SU(2)$ group manifold. Such a map has a winding number

$$N = k \int \mathrm{d}^3 x \epsilon^{ijk} \mathrm{Tr}\, g^{-1}\partial_i g\, g^{-1}\partial_j g\, g^{-1}\partial_k g. \tag{6}$$

(b) Show that N is unchanged if g is wiggled slightly, say by $g \to g(1+if)$ where you work to first order in f. (You will have to integrate by parts.) It follows, since you can iterate the operation of wiggling g slightly, that N is invariant if g is deformed in an arbitrary way in its homotopy class. What is the value of k such that N is integer-valued?

A map from S^3 to $SU(2)$ is "homotopic to the identity" if and only if $N = 0$. The importance of this is that the Gauss law constraint of gauge theory says that a gauge transformation acts trivially on quantum states if (i) it is 1 at infinity, and (ii) it is homotopic to the identity.

(c) Explain the claim in the last sentence.

(d) Show that if the magnetic charge is 0, the gauge transformation g defined in eqn. (5) has $N = 0$. (In showing this, you may assume that zero magnetic charge means that at infinity, you can take ϕ to be constant, that is, to be everywhere in the form (4). Using this, show that $N = 0$.)

The θ angle of gauge theory enters the Hamiltonian formalism in the following way. Let T be a gauge transformation that is 1 at infinity and has $N = 1$. Then in acting in physical states,

$$T = \exp(i\theta), \tag{7}$$

where θ is the gauge theory θ angle. It does not matter precisely what T we use, since if T' is another gauge transformation with the same properties as T (it is 1 at infinity and has $N = 1$), then $T' = gT$ where g is 1 at infinity and has $N = 0$. So $g = 1$ in acting on physical states of gauge theory and T' is equivalent to T.

A generalization of (7) is that if W is any gauge transformation of any winding number N, equal to 1 at infinity, then acting on physical states,

$$W = \exp(iN\theta). \tag{8}$$

(e) Make sure you understand this explanation of what the θ angle means.[1] Show that (8) only makes sense if winding number adds when we multiply gauge transformations, that is we need to know the following: if W_1 has winding number N_1 and W_2 has winding number N_2, then the winding number of the product $W = W_1 W_2$ is

$$N = N_1 + N_2. \tag{9}$$

This is a standard fact; can you prove it from the formula (6)?

Conversely, if you know (9) and (7), then (8) follows, since T^N has winding number N and clearly obeys (8), and moreover any gauge transformation of winding number N is homotopic to T^N.

Now, let us go back to our problem of the magnetic monopole.

(f) Show that if the magnetic charge is 1, that is if ϕ looks like (2) near infinity, then, g has $N = 2$.

You can do this exercise by using the definition (6) of N, but there actually is a faster way. $N = 1$ for a gauge transformation g_0 such that every point on the $SU(2)$ manifold is $g_0(\vec{x})$ for a unique x. Actually, this criterion only ensures that $N = \pm 1$. If g_0 has the stated property, then so does g_0^{-1}, and N is odd under $g_0 \to g_0^{-1}$. (To distinguish $N = 1$ from $N = -1$, one takes an arbitrary value of $\vec{x}$ at which the three elements of the Lie algebra of $SU(2)$ given by the spatial derivatives $a_i = g_0^{-1} \mathrm{d}g_0/\mathrm{d}x_i$, $i = 1, 2, 3$ are linearly independent. Then, the sign of N is the sign of $\mathrm{Tr}\, a_1 a_2 a_3$. There is a general criterion to compute N for any $g(\vec{x})$ that generalizes the criterion just stated if $N = \pm 1$: one counts the points $\vec{x}$ with $g_0(\vec{x}) = b$, for a generic $b \in SU(2)$, and weights each such point with the sign of $\mathrm{Tr}\, a_1 a_2 a_3$.) For the present purposes, don't worry too much about the sign as we have not been very precise with all of our sign conventions. So:

(f') Show that if ϕ is everywhere of the form

$$\phi \to \frac{f(r)}{r} \vec{\sigma} \cdot \vec{x}, \tag{10}$$

[1] In the lecture, I will follow a slightly different, but also standard, approach in which the quantum wavefunction is invariant under all gauge transformations, with $N = 0$ or not, and θ appears as the coefficient of an interaction in the Lagrangian. We might call this the Lagrangian approach to the θ angle. In these notes, I use an alternative Hamiltonian approach in which θ is incorporated as the phase by which a topologically nontrivial gauge transformation acts on the quantum states. It is best to be familiar with both of these (standard) points of view.

then

$$g_0(\vec{x}) = \exp(i\pi\varepsilon) \tag{11}$$

assumes every value on the $SU(2)$ manifold precisely once. (One includes the value at $\vec{x} = \infty$.) Note that g_0 is *not* the same as g. It is defined with an extra 1/2 in the exponent, as if we were normalizing the electric charge so that a W boson (rather than a quark) has unit electric charge; hence $g_0 \to -1$ at infinity, rather than $+1$. Still, g_0 is constant at infinity so the definition of winding number and all our statements about it make sense. Since it assumes every value once, g_0 has winding number $N = 1$. As $g = g_0^2$, deduce from (9) that g has $N = 2$.

Finally, from (8), we learn that the action of g on a physical state of magnetic charge 1 is

$$g = \exp(2i\theta). \tag{12}$$

Since $g = \exp(2\pi i Q_e)$, we learn that magnetic monopoles (of monopole charge 1) obey

$$\exp(2\pi i Q_e) = \exp(2i\theta), \tag{13}$$

so that the electic charge of such a monopole is an integer plus $2\theta/2\pi$. Thus, under $\theta \to \theta + 2\pi$, the electric charge of a monopole is shifted by the charge of a W boson.

Let us now ask what is the complete electric charge spectrum of magnetic monopoles at $\theta = 0$. Let us first assume that at $\theta = 0$, one of the possible values of the electric charge of a magnetic monopole is $Q_e = 0$.[2] By shifting θ by 2π, find that at $\theta = 2\pi$, the theory has a monopole of magnetic charge 1 and electric charge 2 (that is, electric charge equal to that of a W boson). On the other hand, physics is invariant under shifting θ by 2π. That is why it is called a theta *angle*. So the theory at $\theta = 0$ must have a monopole with W boson charge as well as one with electric charge 0. Repeating this process, we see that at $\theta = 0$, a monopole may have an electric charge equal to any integer multiple of the charge of a W boson.

Explicit semiclassical quantization gives this answer. Now let us ask: why isn't there also, at the same value of θ, a monopole with half the charge of a W boson, that is, a monopole with the electric charge of a quark? Such

[2]This almost follows from CP, which reverses the sign of electric charge without reversing the sign of magnetic charge. There is one subtlety: CP would allow the values of Q_e for a magnetic monopole to be either all integers or all half-integers. For here, we will not worry about this; we simply define $\theta = 0$ (as opposed to $\theta = \pi$, which is also CP conserving) to be the value of θ at which a monopole can be electrically neutral.

a monopole could annihilate with an antimonopole of electric charge zero to make a magnetically neutral particle of electric charge half that of a W boson. This would contradict the fact that in the present theory, as there are no quark fields, the W boson charge is the basic unit of electric charge.