Maldacena

1.4 We evaluate the action

$$S_E = -\frac{R^2_{AdS}}{16\pi G_N}\left[\int_{\Sigma_4}\sqrt{g}(R+6) + 2\int_{\partial\Sigma_4}\sqrt{h}K\right]$$

onshell for the "AdS" metric

$$ds^2 = d\rho^2 + \sinh^2\rho \; d\Omega_3^2$$

where the boundary is taken to be at S^3, as $\rho \to \infty$.

The scalar curvature for a space of constant curvature is given by $R = 4\Lambda$. Here, $\Lambda = -3$, so $R = -12$.

The extrinsic curvature K is given by

$$K = \frac{1}{2}h^{ab}\partial_n h_{ab} = \frac{1}{2}h^{-1}\partial_n h = \frac{1}{2}\sinh^{-6}\rho\;\partial_\rho(\sinh^{-6}\rho)$$

$$= -3\frac{\cosh\rho}{\sinh\rho}$$

So the action is

$$S_E = -\frac{R^2_{AdS}}{16\pi G_N}\left[\int_{\Sigma_4}\sqrt{g}\times -6 - 6\int_{\partial\Sigma_4}\sqrt{h}\frac{\cosh\rho}{\sinh\rho}\right]$$

1.4 (Cont)

$$= -\frac{R_{AdS}^2}{16\pi G_N} Vol(S_3)\left[-6\int_0^{\rho_c} \sinh^3\rho \, d\rho + 6\frac{\cosh\rho_c}{\sinh\rho_c}\sinh^3\rho_c\right]$$

$$= -\frac{R_{AdS}^2}{16\pi G_N} 2\pi^2\left[6\left[\frac{1}{3}\cosh^3\rho - \cosh\rho\right]_0^{\rho_c} + 6\cosh\rho_c \sinh^2\rho_c\right]$$

(keeping only finite terms)

$$= -\frac{R_{AdS}^2}{16\pi G_N} \times 2\pi^2 \times +6\left[\frac{1}{3} - 1\right] = \frac{R_{AdS}^2\pi}{2G_N}$$

The path integral $Z = \int d[\phi]\, e^{-S_E[\phi]}$ is taken over all field configuration ; however the dominant contribution comes from where S_E is minimised (other solutions are exponentially suppressed). However, this corresponds to a classical solution.

Therefore the wavefunction for AdS_4 can be approximated by

$$\Psi = Z \sim e^{-S_E} = \exp\left(-\frac{R_{AdS}^2\pi}{2G_N}\right)$$

Exercises on Inflation (Creminelli)

Group 10

Consider the problem of evaluating the n-point correlation functions of a scalar field on a fixed de Sitter background. We work with the metric

$$ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2}\,. \tag{1}$$

It is possible to obtain the functional dependence of the correlation functions on the spacetime coordinates using symmetry arguments. Suppose we perform the following scaling on our spacetime:

$$\eta \to \lambda\eta \quad \text{and} \quad x \to \lambda x\,. \tag{2}$$

This coordinate transformation is an isometry, so any calculation or prediction made on a de Sitter background (in these coordinates) should respect this symmetry; i.e. at the end of a calculation we can perform the above scaling, and the results will still hold.

Question 1

We wish to calculate the two point correlator for a massive scalar field, where the mass is small:

$$m^2 \ll H\,. \tag{3}$$

The fourier transform is defined as

$$f_{\vec{k}} = \int dx^3\, f(\vec{x})e^{-i\vec{x}\cdot\vec{k}} \tag{4}$$

so under the above scaling, $k \to \lambda^{-1}k$ and $f_{\vec{k}} \to \lambda^3 f_{\lambda^{-1}\vec{k}}$. Therefore the two point correlator scales as

$$\langle\phi_{\vec{k_1}}(\eta)\phi_{\vec{k_2}}(\eta)\rangle \to \lambda^6\langle\phi_{\lambda^{-3}\vec{k_1}}(\lambda\eta)\phi_{\lambda^{-3}\vec{k_2}}(\lambda\eta)\rangle\,. \tag{5}$$

Therefore, up to some overall constant, the functional dependence of the two point correlator on the spacetime coordinates is given by

$$\langle\phi_{\vec{k_1}}(\eta)\phi_{\vec{k_2}}(\eta)\rangle \sim \delta^3(\vec{k_1}+\vec{k_2})\sum_p \eta^p k^{-3+p} = \delta^3((\vec{k_1}+\vec{k_2})\frac{F(k\eta)}{k^3} \tag{6}$$

since $\delta^{(3)} \to \lambda^3\delta^{(3)}$. The exponents p can take any values; only the above combination of powers of k and η will give the correct scaling behaviour of the correlator. We can succintly write this condition by absorbing the terms into a general function $F(k\eta)$, which in particular has trivial scaling. Now, by studying the late time behaviour of the scalar perturbations, we will fix the functional dependence of the correlator on η, and hence k. To do this, we simply solve the equations of motion. The equation of motion for a massive scalar field on a fixed de Sitter background (for the above metric) is given by

$$\ddot{\phi} - \frac{2}{\eta}\dot{\phi} + \frac{m^2}{H^2\eta^2}\phi = 0 \tag{7}$$

where we have neglected the gradient term. This admits a power law solution of the form $\phi = \eta^s$; plugging in this ansatz, we get the relation

$$s(s-1) - 2s + \frac{m^2}{H^2} = s^2 - 3s + \frac{m^2}{H^2} = 0\,. \tag{8}$$

The solutions to this quadratic equaiton are given by

$$s = \frac{3 \pm \sqrt{9 - 4\frac{m^2}{H^2}}}{2}\,. \tag{9}$$

We are working in the $m^2 \ll H^2$ regime[1]. Both give rise to decaying solutions, but the minus solution decays slower than the plus solution, so we will only consider the contribution from the minus solution. We have now established that the time dependence of the scalar field solution is given by

$$\phi_{\vec{k}} \sim \eta^{\frac{3-\sqrt{9-4\frac{m^2}{H^2}}}{2}} \tag{10}$$

so therefore by the analysis carried out above, the functional form of the 2 point correlator is given by

$$\langle \phi_{\vec{k_1}}(\eta)\phi_{\vec{k_2}}(\eta)\rangle \sim \delta^3(\vec{k_1} + \vec{k_2})\frac{(k\eta)^{3-\sqrt{9-4\frac{m^2}{H^2}}}}{k^3}\,. \tag{11}$$

So the k dependence of the correlator is given by

$$\langle \phi_{\vec{k_1}}(\eta)\phi_{\vec{k_2}}(\eta)\rangle \sim k^{-\sqrt{9-4\frac{m^2}{H^2}}}\,. \tag{12}$$

and hence the spectral index (tilt) is

$$n_s - 1 = \frac{d\log P_\phi}{d\log k} = 3 - \sqrt{9 - 4\frac{m^2}{H^2}} \approx \frac{2m^2}{3H^2}\,. \tag{13}$$

[1]If $\frac{m^2}{H^2} > \frac{9}{4}$ then even at late times we will get oscillatory solutions.

Question 2

For the general n-point correlator for a (massless) scalar field, the analysis is almost identical. Now the correlator scales as

$$\langle \phi_{\vec{k_1}}(\eta) \dots \phi_{\vec{k_n}}(\eta) \rangle \to \lambda^{3n} \langle \phi_{\lambda^{-1}\vec{k_1}}(\lambda\eta) \dots \phi_{\lambda^{-1}\vec{k_n}}(\lambda\eta) \rangle \,. \tag{14}$$

For a massless scalar, from the analysis carried out above on the equations of motion, we see that the dominant solution is given by the $\phi \sim \eta^0$ solution - so the n-point correlator for a massless field is time independent on sufficiently large scales. Therefore, the functional form of the correlator on superhorizon scales will be given by

$$\langle \phi_{\vec{k_1}}(\eta) \dots \phi_{\vec{k_n}}(\eta) \rangle \sim \delta^{(3)}(\vec{k_1} + \dots + \vec{k_n}) F(k_i) \,. \tag{15}$$

As with before the delta function scales like λ^3, so therefore the function $F(k_i)$ must scale like λ^{3n-3}, i.e. F is a homogeneous function of the k_i of degree $-3(n-1)$.

Group 10.

Creminelli 3.

$$\langle \varphi(x,\eta)\,\varphi(0,\eta)\rangle$$
$$= \frac{1}{(2\pi)^3}\int d^3\vec{k}\,\frac{1}{|\vec{k}|^3}\left(1+|\vec{k}|^2\eta^2\right)e^{i\vec{k}\cdot\vec{x}}$$
$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty\int_0^\pi e^{i|\vec{k}|x\cos\theta}\,\frac{1}{|\vec{k}|}\left(1+k^2\eta^2\right)\sin\theta\,d\theta\,d|\vec{k}|$$
$$= \frac{1}{(2\pi)^2}\int_0^\infty\int_{-1}^1 \frac{1}{i|\vec{k}|x}\,e^{i|\vec{k}|xz}\,\frac{1}{|\vec{k}|}\left(1+|\vec{k}|^2\eta^2\right)dz\,d|\vec{k}|$$
$$= \frac{1}{(2\pi)^2}\int_0^\infty \frac{1}{i|\vec{k}|^2x}\left(1+|\vec{k}|^2\eta^2\right)\left(e^{i|\vec{k}|x}-e^{-i|\vec{k}|x}\right)d|\vec{k}|$$
$$= \frac{1}{(2\pi)^2}\,\frac{1}{ix}\int_0^\infty \frac{1}{|\vec{k}|^2}\left(1+|\vec{k}|^2\eta^2\right)2i\,\sin(|\vec{k}|x)\,d|\vec{k}|$$
$$= \frac{1}{2\pi^2}\,\frac{1}{x}\int_0^\infty \frac{1}{|\vec{k}|^2}\left(1+|\vec{k}|^2\eta^2\right)\sin(|\vec{k}|x)\,d|\vec{k}|$$

Mathematica

$$\overset{\downarrow}{=}\ \frac{1}{2\pi^2}\,\frac{1}{x}\,\frac{1}{x}\left[-x^2\log x-(\gamma-1)x^2+\eta^2\right]$$
$$= \frac{1}{2\pi^2}\cdot\left[-\log x-(\gamma-1)+\frac{\eta^2}{x^2}\right]$$

where $\gamma = 0.5772\ldots$

Explanation: IR diverge happens when adding all the modes.
Or alternatively, it is similar to the soft photon divergence in QED.
Or, we can explain it using random walk, in which

$$\langle\delta\varphi^2\rangle \sim H^3 t \sim H^3\log x$$

Susskind Problem 4 Group 10

4.4 probability to be in vacuum a after n+1 steps

$$P_a(n+1) = P_a(n) + \Sigma_b(-\gamma_{ba})P_a(n) + \Sigma_b \gamma_{ab} P_b(n)$$

where γ_{ba} is transition coeff. for transition from a to b

let $\gamma_{ba} = M_{ba} e^{S_b}$ where $M_{ba} = M_{ab}$

$P_a = e^{S_a}\phi_a$

$$P_a(n+1) = P_a(n) + \Sigma_b(-M_{ba}e^{S_b})e^{S_a}\phi_a(n) + \Sigma_b(M_{ab}e^{S_a})e^{S_b}\phi_b(n)$$

$$P_a(n+1) - P_a(n) = \Sigma_b e^{S_a+S_b}(M_{ab}\phi_b(n) - M_{ba}\phi_a(n))$$

$$= \Sigma_b e^{S_a+S_b} M_{ab}(\phi_b(n) - \phi_a(n))$$

get altered transition matrix:

$$\begin{pmatrix} -\Sigma_{b\neq 1} e^{S_1+S_b}M_{1b} & e^{S_1+S_2}M_{12} & e^{S_1+S_3}M_{13} & \\ e^{S_2+S_1}M_{21} & -\Sigma_{b\neq 2}e^{S_2+S_b}M_{2b} & e^{S_2+S_3}M_{23} & \\ e^{S_3+S_1}M_{31} & e^{S_3+S_2}M_{32} & -\Sigma_{b\neq 3}e^{S_3+S_b}M_{3b} & \\ & & & \ddots \end{pmatrix} \equiv \tilde{M}$$

matrix is symmetric since M_{ab} symmetric and $e^{S_a+S_b} = e^{S_b+S_a}$

$$P(n+1) - P(n) = \tilde{M}\phi(n) \qquad \text{where } P(n) = \begin{pmatrix} P_1(n) \\ P_2(n) \\ \vdots \end{pmatrix} \quad \phi(n) = \begin{pmatrix} \phi_1(n) \\ \phi_2(n) \\ \vdots \end{pmatrix}$$

obvious eigenvalue 0 for eigenvector $\phi(n) \sim \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}$

since diagonal terms contain the negative sum of rest of entries in their rows

$\tilde{M}$ diagonalizable since $\tilde{M}$ symmetric w/ real entries

$\mathrm{Tr}(\tilde{M})$ = sum of eigenvalues $= -\Sigma_a \Sigma_{b\neq a} e^{S_a+S_b} M_{ab} < 0$

Group 10

① The bias function is given by $b = M \frac{d n_g}{d \rho_m}$ where ρ_m includes both the homogeneous background matter density as well as the long wavelength matter density perturbation i.e. $\rho_m = \rho_0 + \delta\rho_m$.

By the chain rule,

$$b = M \frac{d n_g}{d F} \frac{d F}{d \rho_m} = \rho_0 \frac{d F}{d(\delta\rho_m)} = \frac{d F}{d \delta} \qquad \left(\text{since } \delta = \frac{\delta\rho_m}{\rho_0}\right)$$

where

$$F(\delta > \delta_c \mid \delta_p) = \int_{\delta_c - \delta_p}^{\infty} d\delta \exp\left(-\frac{\delta^2}{2\sigma^2(m)}\right)$$

$$\Rightarrow b = \exp\left(-\frac{(\delta_c - \delta_p)^2}{2\sigma^2(m)}\right)$$