# Maldacena Problem 1.4

Group 16

We want to compute

$$\frac{\mathrm{R}_{\mathrm{AdS}}^2}{16\pi G_N}\left[-\int\sqrt{g}(R+6)-2\int K\right] \tag{1}$$

with the metric,

$$\mathrm{d}s^2=\mathrm{d}\rho^2+\sinh^2(\rho)\mathrm{d}\rho_3^2$$

upto $\rho_c$

$$K=\frac{1}{2}h^{ab}\partial_\rho h_{ab}|_{\rho_c}=\frac{3}{2}\frac{2\cosh(\rho_c)}{\sinh(\rho_c)}=2\coth(\rho_c) \tag{2}$$

Einstein's equation in vacuum for the metric takes the following form,

$$\mathrm{R}_{\mu\nu}-\frac{\mathrm{R}}{2}g_{\mu\nu}=-\Lambda g_{\mu\nu}$$

with cosmological constant being$\Lambda=-3$
taking the trace we havs

$$\mathrm{R}-2\mathrm{R}=12$$

and we have

$$\mathrm{R}=-12 \tag{3}$$

therefore,

$$
\begin{aligned}
S &= \frac{\mathrm{R}_{\mathrm{AdS}}^2}{16\pi G_N}\left[-\int \sqrt{g}(-6) - 2\int 3\coth(\rho_c)\right] \\
&= 2\pi^2 \frac{\mathrm{R}_{\mathrm{AdS}}^2}{16\pi G_N}\left[6\int_0^{\rho_c} \sinh^3\rho \mathrm{d}\rho - 6\cosh(\rho_c)\sinh^2(\rho_c)\right] \\
&= 2\pi^2 \frac{\mathrm{R}_{\mathrm{AdS}}^2}{16\pi G_N}\left[6\left[\frac{1}{3}\cosh^3(\rho_c) - \cosh(\rho_c) + \frac{2}{3}\right] - 6\cosh(\rho_c)\sinh^2(\rho_c)\right] \\
&= \frac{\pi R_{AdS}^2}{2G_N} + \text{divergent term}
\end{aligned} \tag{4}
$$

The first term in the action computed in the above equation is the finite part.

Group 16
Problem Solutions
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# 1 Creminelli Problem 3

Calculate the equal time 2-point function of a massless scalar in a fixed de Sitter background in real space. What is the physical meaning of the IR divergence?

$$ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2} \tag{1}$$

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}g^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{2}\int d^4x\frac{\dot{\phi}^2 - (\nabla\phi)^2}{\eta^2} \tag{2}$$

$$\phi = \int d^3x e^{i\vec{k}\cdot\vec{x}}\phi_k(\eta) \tag{3}$$

Wave equation:

$$\frac{\partial}{\partial\eta}\frac{2\dot{\phi}_k}{\eta^2} + \frac{2k^2}{\eta^2}\phi_k^2 = 0 \tag{4}$$

$$\frac{2\ddot{\phi}_k}{\eta^2} - \frac{4\dot{\phi}_k}{\eta^3} + \frac{2k^2}{\eta^2}\phi_k^2 = 0 \tag{5}$$

$$\ddot{\phi}_k - \frac{2\dot{\phi}_k}{\eta} + k^2\phi_k^2 = 0 \tag{6}$$

Solution of the form:

$$f \sim (1 + i \mid k \mid \eta)e^{-i|k|\eta} \tag{7}$$

$$\phi_k = a^\dagger C(1 + i \mid k \mid \eta)e^{-i|k|\eta} + aC(1 - i \mid k \mid \eta)e^{i|k|\eta} \tag{8}$$

Normalize by:

$$[\pi_k, \phi_k] = i \tag{9}$$

$$\pi_k = \frac{\dot{\phi}_k}{\eta^2} \tag{10}$$

$$\pi_k = \frac{1}{\eta^2}C\{a^\dagger[i \mid k \mid e^{-i|k|\eta} - i \mid k \mid (1+i \mid k \mid \eta)e^{-i|k|\eta}] + a[-i \mid k \mid e^{i|k|\eta} + -i \mid k \mid (1-i \mid k \mid \eta)e^{i|k|\eta}]\} \tag{11}$$

$$\pi_k = \frac{1}{\eta^2} C[a^\dagger \mid k \mid^2 \eta e^{-i|k|\eta} + a[\mid k \mid^2 \eta e^{i|k|\eta}] \tag{12}$$

$$\begin{aligned}
[\pi_k, \phi_k] = & - \frac{1}{\eta^2} C^2 [aa^\dagger (1 + i \mid k \mid \eta) \mid k \mid^2 \eta + aa^\dagger (1 - i \mid k \mid \eta) \mid k \mid^2 \eta \\
& - a^\dagger a \mid k \mid^2 \eta (1 - i \mid k \mid \eta) - aa^\dagger \mid k \mid^2 \eta (1 + i \mid k \mid \eta)]
\end{aligned} \tag{13}$$

$$\begin{aligned}
[\pi_k, \phi_k] &= -\frac{1}{\eta^2} C^2 [- \mid k \mid^2 \eta (1 + i \mid k \mid \eta) + \mid k \mid^2 \eta (1 - i \mid k \mid \eta)] \\
&= 2i \mid k \mid^3 C^2
\end{aligned} \tag{14}$$

$$C = \frac{1}{\sqrt{2k^3}} \tag{15}$$

The 2-point function in Fourier Space:

$$\begin{aligned}
< \phi_{\vec{k}}(\eta) \phi_{\vec{k\prime}}(\eta) > &= f^* f \ \ < BD \mid aa^\dagger \mid BD > \\
&= \frac{1}{2k^3} (1 - i \mid k \mid \eta)(1 + i \mid k \mid \eta) \\
&= \frac{1}{2k^3} (1 + \mid k \mid^2 \eta^2)
\end{aligned} \tag{16}$$

In Real Space:

$$\begin{aligned}
\int_\epsilon^\infty \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} (1 + k^2\eta^2) e^{-i\vec{k}\cdot\vec{x}} &= \frac{1}{(2\pi)^2} [\int_\epsilon^\infty \frac{1}{k} \frac{sin(kx)}{kx} dk + \int_\epsilon^\infty k\eta^2 \frac{sin(kx)}{kx} dk] \\
&= \frac{1}{(2\pi)^2} (1 - ln(\epsilon x) + \frac{\eta^2}{k^2 \epsilon^2})
\end{aligned} \tag{17}$$

where there is a ln divergence IR term for $\epsilon \to 0$.
This is different from the massive scalar case in which there is a cutoff at the mass scale. Here the long modes contribute infinitely to the 2-point function, whereas in the massive scalar case they are cut off. This might be an effect of our assumption of the vacuum as the Bunch-Davies vacuum, which is appropriate for modes initially inside the Hubble radius.

# Solution to problem 3.4

The rate equation is

$$\frac{d}{dt}P_a = -\sum_b \Gamma_{ba} P_a + \sum_b \Gamma_{ab} P_b. \tag{1}$$

Make the detailed balance assumption that the matrix $\Gamma$ can be written $\Gamma_{ab} = e^{S_a} M_{ab}$ with $M$ a symmetric matrix. Define $\phi_a = e^{-S_a/2} P_a$. Then the rate equation is

$$\frac{d}{dt}\phi_a = -\sum_b e^{S_b} M_{ba} \phi_a + \sum_b e^{S_a/2} M_{ab} e^{S_b/2} \phi_b. \tag{2}$$

Clearly, the rate equation is now symmetric, since the first term is diagonal and the second is symmetric. It is easy to see that there is a zero eigenvector, corresponding to $\phi_a^{(0)} = e^{S_a/2}$, i.e. $P_a^{(0)} = e^{S_a}$. To check this, note that the two terms on the RHS of (2) are equal and oppostie when evaluated on this vector, so the RHS is zero.

It remains to show that there are no positive eigenvalues. Note that if all $P_a$ start out positive, they remain so. This is because $\Gamma_{ab} \geq 0$, so the only negative parts of the rate equation are diagonal. This means that $\sum_a P_a$ is a sum of positive terms. Thus, if there was a positive eigenvalue, it would eventually grow exponentially. However, $d/dt \sum_a P_a = -\sum_{ab} \Gamma_{ba} P_a + \sum_{ab} P_b = 0$. Thus there are no positive eigenvalues.

# 4. Mechanisms for Inflation (E. Silverstein)

## Problem 1.a)

$$\mathcal{L} = (\partial\phi)^2 - \mu^{4-p}\lambda\phi^p \,, \tag{1}$$

The field equation and Friedmann equation give

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \Rightarrow \dot{\phi} \simeq \tfrac{V'}{3H} \,, \tag{2}$$

$$M_{\rm pl}H^2 = \tfrac{1}{3}(\tfrac{1}{2}\dot{\phi}^2 + V) \Rightarrow 2M_{\rm pl}^2 H\dot{H} = \tfrac{1}{3}(\ddot{\phi} + V') = -H\dot{\phi}^2 \tag{3}$$

$$\Rightarrow M_{\rm pl}^2\dot{H} = -\dot{\phi}^2/2 \,. \tag{4}$$

For the number of e-foldings $N_e$ we get

$$N_e = \int H dt = \int H\frac{d\phi}{\dot{\phi}} \simeq \int \frac{3H^2}{V'}d\phi \simeq \int \frac{V}{M_{\rm pl}^2 V'}d\phi = \frac{1}{2pM_{\rm pl}^2}(\phi_i^2 - \phi_f^2) \,, \tag{5}$$

Where $\phi_i$ and $\phi_f$ are initial and final field values respectively. Equation (5) gives the first condition on the field range from the required number of e-folds.

The second requirement comes from the two-point function of scalar curvature fluctuations. However, it is useful to first express $\phi$ in terms of slow-roll parameter $\epsilon$:

$$\epsilon = -\frac{\dot{H}}{H^2} \simeq \frac{\dot{\phi}^2/2}{V/3} = \frac{V'^2}{6H^2V} \simeq \frac{M_{\rm pl}^2 V'^2}{2V^2} = \frac{p^2 M_{\rm pl}^2}{2\phi^2} \,, \tag{6}$$

Neglecting order one factors the amplitude of scalar fluctuations is

$$\zeta \simeq 10^{-5} \simeq \tfrac{H^2}{\dot{\phi}} \simeq \tfrac{H^3}{V'} \,, \tag{7}$$

$$\Rightarrow \zeta^2 \simeq \left(\tfrac{V}{M_{\rm pl}V'}\right)^2 \tfrac{V}{M_{\rm pl}^4} \simeq \tfrac{\mu^{4-p}\lambda\phi^p}{\epsilon M_{\rm pl}^4} \simeq \tfrac{\lambda\mu^{4-p}}{M_{\rm pl}^{4-p}} p^p \epsilon^{-1-p/2} \tag{8}$$

$$\Rightarrow \tfrac{\mu}{M_{\rm pl}} \sim \left(\tfrac{\epsilon^{1+p/2}\zeta^2}{p^p\lambda}\right)^{1/(4-p)} \,. \tag{9}$$

## b)

The main issue with the theory (1) is to obtain such a small value of $\mu$ as derived in (9). Moreover it should be maintained in a large range of the scalar field $\phi$ in the case of $p \sim 1$. Note that one expects contributions of the form

$$\Delta V \sim \phi^4 \left(\frac{\phi}{M_{\rm pl}}\right)^n \,, \tag{10}$$

to be generically present in the theory and a large field range $\Delta\phi > M_{\rm pl}$ makes these corrections important.

Nevertheless this theory is radiatively stable if we postulate an approximate shift on $\phi$

$$\phi \to \phi + c\,. \tag{11}$$

This symmetry is broken by the potential term. Therefore any radiative correction to the potential of $\phi$ must be proportional to the symmetry breaking parameter $\lambda\mu^{4-p}$. On the other hand since there are no self-interactions of $\phi$ the radiative corrections should come from the loops of gravitons and therefore the relevant quantity to consider is $\mu/M_{\rm pl}$ which is very small as derived in (9).

## c)

Additional constraints arise from UV completion considerations. For instance in small field scenarios obtaining a very small value for $\mu$ needs a delicate cancellation between different terms coming from moduli-stabilizing potential and the repulsive potential of a D3-brane moving in a warped throat.

In large field scenarios one faces the additional constraint that the compactifications usually allow only a finite variation of the scalar fields. Moreover the shift symmetry of the axion fields may be brocken by multiple sources which again require fine-tuning to obtain the desired flat potential.

# 5. CMB and LSS (Zaldariaga)

**Problem 4)**

The distance to last scattering surface determines the position of the peaks. $\Omega_m h^2$ determines matter-radiation equality which then results in the suppression of modes that enter horizon later. $\Omega_b h^2$ determines baryon loading that suppresses even acoustic peaks.

Changing the distance to the LSS changes the position of the peaks. The degeneracy arises because for a fixed distance to LSS we can have different values of spatial curvature by changing Hubble parameter $h$. $\Omega_\Lambda$ and $\Omega_m$ can be modified simultaneously to get the same value for $\Omega_m h^2$ and satisfy $\sum \Omega_i = 1$. Finally one needs to change $\Omega_b$ to keep $\Omega_b h^2$ fixed.

# 7. Stars and Galaxies (Spergel)

**Problem 1)**

In Press-Schechter model the number density of halos per unit mass interval is given by

$$n = \frac{2\bar{\rho}}{M}\frac{\partial}{\partial M}\int_{\delta_c}^{\infty} d\delta_M P(\delta_M)\,, \tag{1}$$

where $\delta_M = \delta\rho_M/\bar{\rho}$ is the averaged fractional density perturbation on scales $M$ and $\delta_c$ is the critical density above which halos form. Moreover $P(\delta_M)$ is given by a Gaussian

$$P(\delta_M) = \frac{e^{-\delta_M^2/2\sigma_M^2}}{\sqrt{2\pi}\sigma_M}\,. \tag{2}$$

Bias is the relation between long wavelength and short wavelength perturbations. However a long wavelength over-density effectively reduces the critical density $\delta_c$ therefore

$$b = \frac{\partial}{\partial\delta_c}\ln n = \delta_c^{-1} - \frac{\delta_c}{\sigma_M^2}\,. \tag{3}$$

## 6. Modified Gravity (Arkani-Hamed)

**Problem 1)**

We first evaluate polarization sum in 4D

$$N_{\mu\nu\alpha\beta} = \epsilon^{++*}_{\mu\nu}\epsilon^{++}_{\alpha\beta} + \epsilon^{--*}_{\mu\nu}\epsilon^{--}_{\alpha\beta}\,. \tag{1}$$

In a reference where the particle is moving in $z$ direction

$$\epsilon^{\pm\pm}_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}(1 \quad \pm i)\,, \tag{2}$$

and the 2-vectors live in $x$-$y$ space (and zero outside). Plugging this back into (1) we get

$$\begin{aligned} N_{\mu\nu\alpha\beta} &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]_{\mu\nu}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]_{\alpha\beta} + c.c \\ &= -\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta} + \tilde{\eta}_{\mu\alpha}\tilde{\eta}_{\nu\beta} + \tilde{\eta}_{\mu\beta}\tilde{\eta}_{\nu\alpha}\,, \end{aligned} \tag{3}$$

with

$$\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu\bar{p}_\nu + p_\nu\bar{p}_\mu}{p.\bar{p}}, \qquad \bar{p}_\mu = (p_0, 0, 0, -p_3)\,. \tag{4}$$

In D dimention there are $D-2$ possible helicities and the tracelessness of $N_{\mu\nu\alpha\beta}$ leads to

$$N_{\mu\nu\alpha\beta} = -\frac{2}{D-2}\tilde{\eta}_{\mu\nu}\tilde{\eta}_{\alpha\beta} + \tilde{\eta}_{\mu\alpha}\tilde{\eta}_{\nu\beta} + \tilde{\eta}_{\mu\beta}\tilde{\eta}_{\nu\alpha}\,. \tag{5}$$