Group 17 PITP homework

Problem 1.5

The goal of this problem is to find the cutoff-dependence of the Hartle-Hawking wave function for dS space. To leading order in the semiclassical expansion we can approximate this as

$$\Psi[\hat{g}_{ij}] \approx e^{-S_E[g_{cl}]},$$

where g_{cl} is a compact solution of the Euclidean Einstein equations with positive cosmological constant on a manifold with a spherical boundary with induced boundary metric $\hat{g}_{ij}$. The relevant Euclidean action is

$$S_E = \frac{1}{2\kappa^2}\int_M d^4x\sqrt{g}\Big(-R+\frac{6}{\ell_{ds}^2}\Big) - \frac{1}{\kappa^2}\int_{\partial M}\sqrt{\hat{g}}K$$

Here $\kappa^2 = 8\pi G = M_p^{-2}$, and K is the trace of the extrinsic curvature at the boundary. From now on we choose units where $\ell_{ds} = 1$. For generic $\hat{g}_{ij}$ this computation is quite difficult, but we can do it easily in "mini-superspace", where we consider only metrics of the form

$$ds^2 = d\theta^2 + a(\theta)^2 d\Omega_3^2.$$

We are interested in finding solutions where the induced metric is $a_c^2 d\Omega_3^2$ for an a_c of our choice. The equation of motion for a is

$$(a')^2 = 1 - a^2,$$

which has a variety of solutions. Let us first consider the case of $0 < a_c < 1$. There are two real solutions obeying the desired boundary conditions, both of the form $a(\theta) = \sin\theta$. In one case θ runs from 0 to $\arcsin a_c$ while in the other it runs from 0 to $\pi - \arcsin a_c$. Which solution to consider is a subtle problem, Hartle and Hawking for various reasons choose to include both in the path integral. We will be agnostic and compute both. First for the second solution, observing that for this metric we have $K = 3a'/a$ and $R = 12$, we find the action is

$$S_E = -\frac{3V(S^3)}{\kappa^2}\left[\int_0^{\pi-\arcsin a_c}\sin^3\theta d\theta - a_c^2\sqrt{1-a_c^2}\right]$$

Here $V(S^3) = 2\pi^2$ is the volume of the unit S^3. Evaluating the integral, we find

$$S_E = -\frac{4\pi^2}{\kappa^2}\left[1 + (1-a_c^2)\sqrt{1-a_c^2}\right].$$

If we had studied the other solution we would have just gotten the other branch for the square root. Restoring ℓ_{ds} and continuing to $a > 1$, we find the semiclassical wave function for the two solutions is

$$\Psi[a_c] \approx e^{\frac{4\pi^2\ell_{ds}^2}{\kappa^2}\left[1\pm i(1-a_c^2)\sqrt{a_c^2-1}\right]}$$

Note that we can exchange the contributions of the two solutions depending on which way we continue a_c around $a_c = 1$. The choice of branch amounts to selecting the expanding or contracting part of deSitter. The phase is divergent at large a_c while the real part is finite.

Problem 2.2

The linearized Lagrangian for ζ in a dS background $ds^2 = \frac{-dT^2+dx^2}{T^2}$ is invariant under $x' = \lambda x$, $T' = \lambda T$, $\zeta'(x', T') = \zeta(x, T)$, so ζ has dimension 0 in position space. In momentum space it thus has dimension -3, so $\zeta'_{k'} = \lambda^3\zeta_k$, with $k' = k/\lambda$. Applying this to correlation functions and using the de Sitter invariance of the vacuum, this means

$$\langle\zeta_{\lambda k_1}...\zeta_{\lambda k_n}\rangle = \lambda^{-3n}\langle\zeta_{k_1}...\zeta_{k_n}\rangle. \tag{1}$$

Because of translation invariance $\vec{x} \to \vec{x} + \vec{a}$, we can write the correlator as some function $F(k_i)$ times $\delta^3(\sum_i \vec{k}_i)$. Then demanding equation (1) holds we find that $F(\lambda k_i) = \lambda^{3-3n}F(k_i)$, as desired.

Problem 3.5

For a general landscape with three vacua, the rate equation is

$$\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{pmatrix} = \begin{pmatrix} -\gamma_{21}-\gamma_{31} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & -\gamma_{12}-\gamma_{32} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & -\gamma_{13}-\gamma_{23} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

Imposing detailed balance $\gamma_{ij} = M_{ij} e^{-S_j}$ with $M_{ij} = M_{ji}$ and setting vacuum 1 to be terminal, this simplifies to:

$$\begin{pmatrix} \dot{P}_1 \\ \dot{P}_2 \\ \dot{P}_3 \end{pmatrix} = \begin{pmatrix} 0 & M_{12}e^{-S_2} & M_{13}e^{-S_3} \\ 0 & -(M_{12}+M_{23})e^{-S_2} & M_{23}e^{-S_3} \\ 0 & M_{23}e^{-S_2} & -(M_{13}+M_{23})e^{-S_3} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

This has an obvious eigenvector $(1,0,0)$ with eigenvalue 0. The nonzero eigenvalues λ obey an equation

$$\left(\lambda + (M_{12}+M_{23})e^{-S_2}\right)\left(\lambda + (M_{13}+M_{23})e^{-S_3}\right) = (M_{23})^2 e^{-S_2-S_3}.$$

This equation is quadratic, its roots are rather unpleasant but showing that they are negative and real amounts to showing

$$0 < 4e^{-S_2-S_3}(M_{12}M_{13} + M_{12}M_{23} + M_{13}M_{23}) < \left((M_{12}+M_{23})e^{-S_2} + (M_{13}+M_{23})e^{-S_3}\right)^2$$

The first inequality is trivial and the second follows from $4ab < (a+b)^2$ for any real a, b. Of course the reality of the eigenvalues follows from the fact that M_{ij} is symmetric, but it is nice to check it explicitly. It is clear that any initial probablity distribution with a nonzero component along the zero eigenvector will quickly become dominated by the terminal vacuum, so the probability in the other two states must decrease by conservation of probability. So far this has been about a particular fixed physical volume of space, if we embed these rates into a colored Mandelbrot model then whether the total volume in vacua 2 and 3 grows or not depends on whether or not their expansion rates are sufficiently large compared to their decay rates.

Problem 4.1

a. The number of e-folds before the end of inflation is defined as:

$$N = \int_{\phi_e}^{\phi} \frac{da}{a} = \int_{\phi_e}^{\phi} \frac{H}{\dot{\phi}} d\phi$$

Using the slow roll conditions:

$$H^2 = \frac{1}{3M_p^2} V(\phi)$$
$$3H\dot{\phi} + V'(\phi) = 0$$

The number of efolds can be written in terms of the potential:

$$N = \int_{\phi_e}^{\phi} \frac{V(\phi)}{V'(\phi)} d\phi$$

The end of inflation corresponds to $\epsilon = 1$. In the slow roll limit

$$\epsilon \simeq \frac{M_p^2}{2} \left(\frac{V'(\phi)}{V(\phi)}\right)^2$$

Therefore the field value at the end of inflation, ϕ_e, is given by:

$$\phi_e = \frac{pM_p}{\sqrt{2}}$$

So for the potential $V(\phi) = \lambda\mu^{4-p}\phi^p$ the number of efolds is

$$N = \frac{1}{pM_p^2}\int_{\phi_e}^{\phi}\phi d\phi = \frac{1}{2pM_p^2}\phi^2 - \frac{p}{2}$$

The field value -efolds before the end of inflation is then given by:

$$\frac{\phi_N}{M_p} = \sqrt{2p(N + \frac{p}{2})}$$

by using the COBE normalization we can obtain the constraint on μ

$$\Delta^2_{scalar} \simeq \left(\frac{H}{\dot{\phi}}\right)^2 H^2 \simeq \frac{V(\phi)^3}{V'(\phi)^2}|_{\phi=\phi_N} \simeq 10^{-10}$$

Therefore

$$(\frac{\mu}{M_p})^{2-p/2}(\frac{\phi_N}{M_p})^{p/2+1} \simeq 10^{-5}$$

for quadratic potential: $\mu/M_p \simeq 10^{-6}$ and for linear potential: $\mu/M_p \simeq 10^{-3}$.

b. To study quantum corrections to the potential $\mu^{4-p}\phi^p$, we can write $\phi = \phi_0 + \delta\phi$ and observe that the interactions for $\delta\phi$ are suppressed by powers of $\left(\frac{\mu}{\phi_0}\right)^{4-p}$. This quantity goes into all interaction vertices that can renormalize the effective potential, so such renormalizations will be suppressed beyond what we naively expect in effective field theory. Morally this is because there is an approximate shift symmetry which makes such corrections technically natural.

c. In an inflation model coming from string theory, the moduli have been stabilized in various ways which can be overwhelmed if the field value becomes too large.