PITP 2011 Week 1 Problems - Group 19

I. MALDACENA 2

We wish to calculate the three-point function:

$$\begin{aligned}
&\langle BD|\phi_{\mathbf{k}_1}(\eta)\phi_{\mathbf{k}_2}(\eta)\phi_{\mathbf{k}_3}(\eta)|BD\rangle \\
&\qquad = \langle BD|U_{int}^{-1}\left(\eta,-\infty(1-i\epsilon)\right)\phi^I_{\mathbf{k}_1}(\eta)\phi^I_{\mathbf{k}_2}(\eta)\phi^I_{\mathbf{k}_3}(\eta)U_{int}\left(\eta,-\infty(1+i\epsilon)\right)|BD\rangle
\end{aligned}$$

where the superscript I indicates a field in the interaction picture, and we have:

$$U_{int}(\eta,\eta_0) \equiv T\mathrm{exp}\left(-i\int_{\eta_0}^{\eta}\frac{d\eta'}{\eta'}H_{int}(\eta')\right)$$

and the interaction Hamiltonian in the interaction picture is:

$$\begin{aligned}
H_{int}(\eta') &= \int\frac{d^3x}{\eta'^3}\frac{\lambda}{3!}(\phi^I(\mathbf{x},\eta'))^3 \\
&= \frac{\lambda}{3!}\int\frac{d^3x}{\eta'^3}\int d^3k'_1\int d^3k'_2\int d^3k'_3\, e^{i(\mathbf{k'}_1+\mathbf{k'}_2+\mathbf{k'}_3)\cdot\mathbf{x}}\phi^I_{k'_1}(\eta')\phi^I_{k'_2}(\eta')\phi^I_{k'_3}(\eta')
\end{aligned}$$

Expanding to first order in λ, the three point function takes the form:

$$\begin{aligned}
&\langle BD|\phi_{\mathbf{k}_1}(\eta)\phi_{\mathbf{k}_2}(\eta)\phi_{\mathbf{k}_3}(\eta)|BD\rangle \\
&\quad = \frac{\lambda}{3!}\int d^3x\int d^3k'_1\int d^3k'_2\int d^3k'_3\, e^{i(\mathbf{k'}_1+\mathbf{k'}_2+\mathbf{k'}_3)\cdot\mathbf{x}} \\
&\quad \times\langle BD|\Bigg[i\left(\bar{T}\int_{-\infty(1-i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}\phi^I_{\mathbf{k'}_1}(\eta')\phi^I_{\mathbf{k'}_2}(\eta')\phi^I_{\mathbf{k'}_3}(\eta')\right)\phi^I_{\mathbf{k}_1}(\eta)\phi^I_{\mathbf{k}_2}(\eta)\phi^I_{\mathbf{k}_3}(\eta) \\
&\quad - i\phi^I_{\mathbf{k}_1}(\eta)\phi^I_{\mathbf{k}_2}(\eta)\phi^I_{\mathbf{k}_3}(\eta)\left(T\int_{-\infty(1+i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}\phi^I_{\mathbf{k'}_1}(\eta')\phi^I_{\mathbf{k'}_2}(\eta')\phi^I_{\mathbf{k'}_3}(\eta')\right)\Bigg]|BD\rangle
\end{aligned}$$

Now we recall the form of the interaction picture two-point function from problem 1:

$$\langle BD|\phi^I_{\mathbf{k'}}(\eta')\phi^I_{\mathbf{k}}(\eta)|BD\rangle = \frac{1}{2k^3}\delta^3(\mathbf{k'}+\mathbf{k})(1-ik\eta')(1+ik\eta)e^{ik(\eta'-\eta)}$$

Summing over each pairing of the fields and performing the integrals over $\mathbf{x}$, $\mathbf{k'}_1$, $\mathbf{k'}_2$, and $\mathbf{k'}_3$, gives:

$$
\begin{aligned}
&\langle BD|\phi_{\mathbf{k}_1}(\eta)\phi_{\mathbf{k}_2}(\eta)\phi_{\mathbf{k}_3}(\eta)|BD\rangle \\
&\quad= (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)i\lambda\Bigg[\int_{-\infty(1-i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}\prod_i\frac{1}{2k_i^3}(1-ik_i\eta')(1+ik_i\eta)e^{ik_i(\eta'-\eta)} \\
&\qquad\qquad - \int_{-\infty(1+i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}\prod_i\frac{1}{2k_i^3}(1+ik_i\eta')(1-ik_i\eta)e^{-ik_i(\eta'-\eta)}\Bigg] \\
&\quad= -2(2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\lambda\mathrm{Im}\left[\int_{-\infty(1-i\epsilon)}^{\eta}\frac{d\eta'}{\eta'^4}\prod_i\frac{1}{2k_i^3}(1-ik_i\eta')(1+ik_i\eta)e^{ik_i(\eta'-\eta)}\right] \\
&\quad= (2\pi)^3\delta^3(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\frac{2\lambda}{\prod_i(2k_i^3)}\mathrm{Im}\Bigg[\frac{1}{3\eta^3}(i-k_1\eta)(i-k_2\eta)(i-k_3\eta) \\
&\qquad \times i\Big[1-i(k_1+k_2+k_3)\eta+(k_1^2+k_2^3+k_3^2-k_1k_2-k_2k_3-k_1k_3)\eta^2 \\
&\qquad\quad + e^{-i(k_1+k_2+k_3)\eta}(k_1^3+k_2^3+k_3^3)\eta^3\mathrm{Ei}\left(i(k_1+k_2+k_3)\eta\right)\Big]\Bigg]
\end{aligned}
$$

where $\mathrm{Ei}(x)$ is the exponential integral:

$$
Ei(x) = \int_{-\infty}^{x}\frac{e^t}{t}dt
$$

Notice that the choice of contour has killed the contribution from early times.

II. CREMINELLI 5

We take the action of the Goldstone boson to be of the form:

$$
S = \int d^4x\sqrt{-g}\left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\frac{\phi^2}{f_a^2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]
$$

Taking an FRW background and expanding around $\phi = 0$, we find:

$$
S = \int d^4x\, a^3(t)\left[\frac{1}{2}\left(\dot{\phi}^2 - \frac{(\partial_i\phi)^2}{a^2(t)}\right) + \frac{1}{2}\frac{\phi^2}{f_a^2}\left(\dot{\phi}^2 - \frac{(\partial_i\phi)^2}{a^2(t)}\right)\right]
$$

To estimate the deviation from non-Gaussianity, we examine:

$$
\frac{\mathcal{L}_4}{\mathcal{L}_2} = \frac{\frac{1}{2}\frac{\phi^2}{f_a^2}\left(\dot{\phi}^2 - \frac{(\partial_i\phi)^2}{a^2(t)}\right)}{\frac{1}{2}\left(\dot{\phi}^2 - \frac{(\partial_i\phi)^2}{a^2(t)}\right)} = \frac{\phi^2}{f_a^2}
$$

During inflation, the Goldstone boson acquires fluctuations on the order of $\frac{H}{2\pi}$, and so we see that:

$$\frac{\mathcal{L}_4}{\mathcal{L}_2} \sim \frac{H^2}{4\pi^2 f_a^2}$$

If $H \gtrsim 4\pi f_a$, we have $\frac{\mathcal{L}_4}{\mathcal{L}_2} \gtrsim \mathcal{O}(1)$, the interactions become important, and the fluctuations are no longer well approximated as Gaussian.

III. CREMINELLI 8

We will model the reheating process by the decay of an oscillating scalar field into radiation. The Friedmann equations can be expressed as:

$$3H^2 = \rho$$
$$\dot{\rho} = -3H(\rho + p)$$

where we have set $m_p = c = 1$. An oscillating scalar field averaged over many cycles has the same equation of state as that of non-relativistic matter, and so the relevant energy densities evolve as:

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma\rho_\phi$$
$$\dot{\rho}_r + 4H\rho_\phi = \Gamma\rho_\phi$$

The number of e-foldings from the onset of oscillations after inflation up to some surface constant energy density is given by integrating the Hubble parameter

$$N(t_*, t_c) = \int_{t_*}^{t_c} H(t)dt$$

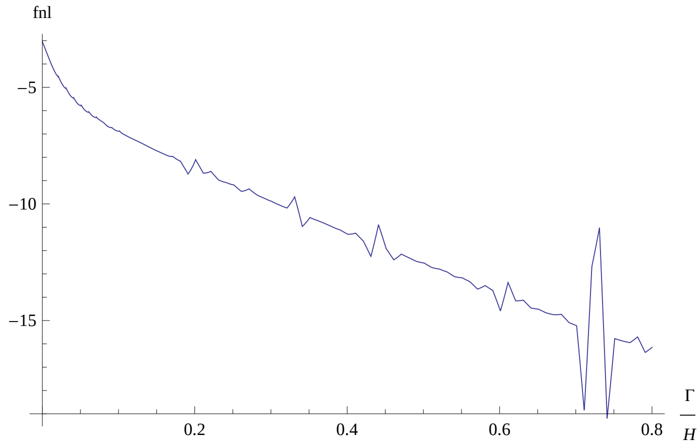
In modulated reheating, the decay rate Γ varies in space, and so we can view the number of e-foldings at some point in space as a function of the decay rate. We can then relate the curvature perturbation ζ to the change in the number of e-foldings as we perturb Γ.

$$\zeta = \delta N = \frac{\partial N}{\partial \Gamma}\delta\Gamma + \frac{1}{2}\frac{\partial^2 N}{\partial \Gamma^2}\delta\Gamma^2 + \dots$$

Taking as the definition of $f_{\rm NL}^{\rm local}$ the relation

$$\zeta = \zeta_g - \frac{3}{5} f_{\rm NL}^{\rm local}\zeta_g^2$$

FIG. 1. $f_{\rm NL}^{\rm local}$ as a function of $\frac{\Gamma}{H_{\rm end}}$



we find that we can compute $f_{\rm NL}^{\rm local}$ by the following formula:

$$f_{\rm NL}^{\rm local} = -\frac{5}{6}\frac{\frac{\partial^2 N}{\partial \Gamma^2}}{\left(\frac{\partial N}{\partial \Gamma}\right)^2}$$

We can numerically solve these equations to find the result for $f_{\rm NL}^{\rm local}$ as a function of $\frac{\Gamma}{H_{\rm end}}$, where $H_{\rm end}$ is the Hubble parameter at the end of inflation, and the result is shown in the figure (the features in the plot are numerical artifacts).

IV. SUSSKIND 2

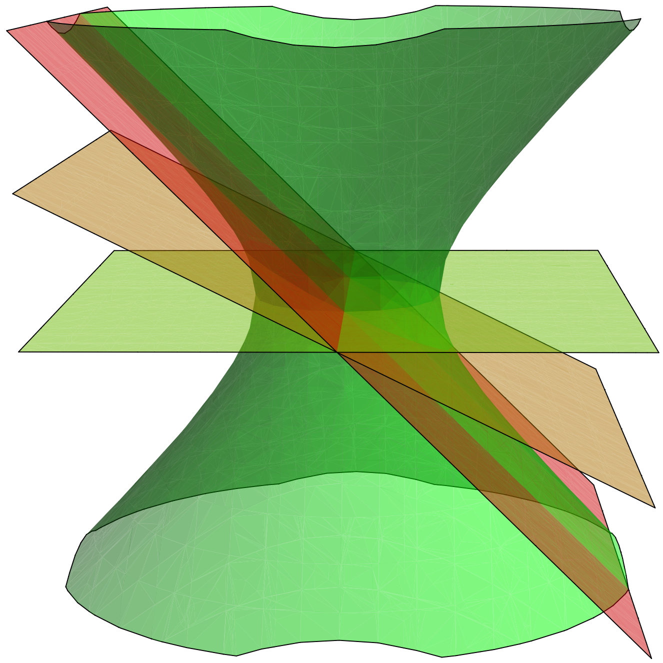
We wish to derive the metric of the static slicing of de Sitter space. We begin by describing de Sitter space as a 4 dimensional hyperboloid embedded in 5 dimensional space.

$$-T^2 + X_i^2 = R^2$$

The static slicing is obtained by the intersection of the hyperboloid with hyperplanes pivoted on the origin of the original coordinate system. The angle of the slice will serve as the time t which runs from $-\pi/2$ to $\pi/2$, and we will also need a radial coordinate r which runs from 0 to the horizon at R.

In figure 2, T runs up and down, perpendicular to the horizontal green plane the coordinate, while X_1 runs left to right, perpendicular to the pivot axis of the planes, and the remaining

FIG. 2. Slicing of a hyperboloid to obtain the static slicing of de Sitter



X_i are the directions orthogonal to X_1. We need to relate the original coordinates to our new set of coordinates t, r, and y_i. The coordinate r is measured perpendicular to the X_1 and T directions, and the worldline corresponding to $r = 0$ is a hyperbola which runs up along the right edge of figure. Points with nonzero r have a worldline corresponding to a hyperbola with a greater eccentricity shifted into or out of the page as compared to the $r = 0$ worldline. Putting things together, we find that the old coordinates are related to the new by the following expressions:

$$
\begin{aligned}
T &= \sqrt{R^2 - r^2}\sinh(t) \\
X_1 &= \sqrt{R^2 - r^2}\cosh(t) \\
X_i &= r y_i
\end{aligned}
$$

It is then a simple matter to find the metric for the static slicing.

$$
\begin{aligned}
dT &= \sqrt{R^2 - r^2}\cosh(t)dt - \frac{r}{\sqrt{R^2 - r^2}}\sinh(t)dr \\
dX_1 &= \sqrt{R^2 - r^2}\sinh(t)dt - \frac{r}{\sqrt{R^2 - r^2}}\cosh(t)dr \\
dX_i &= y_i dr + r dy_i
\end{aligned}
$$

The metric in the new coordinates takes the form:

$$\begin{aligned} ds^2 &= -dT^2 + dX_i^2 \\ &= -(R^2 - r^2)dt^2 + \frac{1}{R^2 - r^2}dr^2 + r^2 d\Omega_2^2 \end{aligned}$$

V. SILVERSTEIN 3

We begin with an action of the form:

$$\begin{aligned} S &= \int d^4x\sqrt{-g}\left[-\frac{\phi^4}{\lambda}\sqrt{1 + \frac{\lambda}{\phi^4}g^{\mu\nu}\partial_\mu\partial_\nu} - V(\phi)\right] \\ &= \int d^4x\sqrt{-g}\left[-\frac{1}{f(\phi)}\sqrt{1 + f(\phi)g^{\mu\nu}\partial_\mu\partial_\nu} - V(\phi)\right] \end{aligned}$$

where we have defined $f(\phi) \equiv \frac{\lambda}{\phi^4}$ to simplify our notation. In an unperturbed FRW universe with $\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$, the unperturbed action takes the form:

$$S_0 = \int d^4x\, a^3\left[-\frac{1}{f(\phi)}\sqrt{1 - f(\phi)\dot{\phi}^2} - V(\phi)\right]$$

We can vary this action to find the equation of motion for ϕ.

$$\begin{aligned} \frac{\delta\mathcal{L}_0}{\delta\dot{\phi}} &= \frac{a^3\dot{\phi}}{\sqrt{1 - f\dot{\phi}^2}} \\ \partial_t\frac{\delta\mathcal{L}_0}{\delta\dot{\phi}} &= a^3\left[\frac{\ddot{\phi}}{\sqrt{1 - f\dot{\phi}^2}} + \frac{\dot{\phi}}{2}\left(1 - f\dot{\phi}^2\right)^{3/2}\left(f'\dot{\phi}^3 + 2f\dot{\phi}\ddot{\phi}\right) + 3H\frac{\dot{\phi}}{\sqrt{1 - f\dot{\phi}^2}}\right] \\ \frac{\delta\mathcal{L}_0}{\delta\phi} &= a^3\left[\frac{f'}{f^2}\sqrt{1 - f(\phi)\dot{\phi}^2} + \frac{f'}{2f}\frac{\dot{\phi}^2}{\sqrt{1 - f(\phi)\dot{\phi}^2}} - V'\right] \end{aligned}$$

If we make the definition $\gamma \equiv \frac{1}{\sqrt{1 - f\dot{\phi}^2}}$, then the equation of motion for ϕ becomes:

$$a^3\left[\gamma\ddot{\phi} + \frac{1}{2}\gamma^3 f'\dot{\phi}^4 + \gamma^3 f\dot{\phi}^3\ddot{\phi} + 3H\gamma\dot{\phi} - \frac{1}{\gamma}\frac{f'}{f} - \frac{\gamma}{2}\frac{f'}{f}\dot{\phi}^2 + V'\right] = 0$$

Using the relation

$$1 + \gamma^2 f\dot{\phi}^2 = 1 + \frac{f\dot{\phi}^2}{1 - f\dot{\phi}^2} = \frac{1}{1 - f\dot{\phi}^2} = \gamma^2$$

we can simplify the equation of motion to obtain:

$$\begin{aligned} \ddot{\phi} + \frac{3}{2}\frac{f'}{f}\dot{\phi}^2 - \frac{f'}{f^2} + 3H\gamma^{-2}\dot{\phi} + \gamma^{-3}V' = 0 \\ \ddot{\phi} - 6\frac{\dot{\phi}^2}{\phi} + \frac{4\phi^3}{\lambda} + 3H\gamma^{-2}\dot{\phi} + \gamma^{-3}V' = 0 \end{aligned}$$

Likewise, we can vary the action for the perturbations:

$$S_2 = \int d^4x\, a^3 \Big[\left(\frac{1}{\sqrt{1-\frac{\phi^4}{\lambda}\dot\phi^2}} + \frac{\phi^4}{\lambda}\dot\phi^2(1-\frac{\phi^4}{\lambda}\dot\phi^2)^{-3/2} \right) \left(\dot{\delta\phi}^2 - a^{-2}(\partial_i\delta\phi)^2 \right)$$
$$+ \left(-20\frac{\lambda}{\phi^6}\sqrt{1-\frac{\phi^4}{\lambda}\dot\phi^2} - \frac{10\dot\phi^2}{\phi^2\sqrt{1-\frac{\phi^4}{\lambda}\dot\phi^2}} + \frac{4\phi^2\dot\phi^4}{\lambda}\left(1-\frac{\lambda}{\phi^4}\dot\phi^2\right)^{-3/2} - V'' \right)\delta\phi^2 \Big]$$

to find the equation of motion for the perturbations

$$\frac{1}{1-\frac{\lambda}{\phi^4}\dot\phi^2}\ddot{\delta\phi} + \frac{3H}{1-\frac{\lambda}{\phi^4}\dot\phi^2}\dot{\delta\phi}\left(\frac{-4\lambda}{\phi^3}\dot\phi^3 + \frac{\lambda}{\phi^4}\dot\phi\ddot\phi\right)\left(1-\frac{\lambda}{\phi^4}\dot\phi^2\right)^{-2}\dot{\delta\phi}$$
$$- \left(-20\frac{\lambda}{\phi^6}\sqrt{1-\frac{\phi^4}{\lambda}\dot\phi^2} - \frac{10\dot\phi^2}{\phi^2\sqrt{1-\frac{\phi^4}{\lambda}\dot\phi^2}} + \frac{4\phi^2\dot\phi^4}{\lambda}\left(1-\frac{\lambda}{\phi^4}\dot\phi^2\right)^{-3/2} - V'' \right)\delta\phi^2$$
$$- \frac{1}{1-\frac{\lambda}{\phi^4}\dot\phi^2}\left(\frac{\partial_i\delta\phi}{a}\right)^2 = 0$$

Now we can look for inflationary solutions. First, we find the energy density and pressure:

$$\rho = \frac{\dot\phi^2}{\sqrt{1-f\dot\phi^2}} + \frac{1}{f}\sqrt{1-f\dot\phi^2} + V = \frac{\gamma}{f} + V$$
$$p = -\frac{1}{f}\sqrt{1-f\dot\phi^2} - V = -\frac{1}{\gamma f} - V = -\rho + \gamma\dot\phi^2$$

Accelerated expansion occurs when $\frac{\ddot a}{a} > 0$, or equivalently when $w \equiv \frac{p}{\rho} < -\frac{1}{3}$. In this case we have:

$$w = \frac{p}{\rho} = -1 + \frac{\gamma\dot\phi}{\frac{\gamma}{f} + V} = -1 + \frac{f\dot\phi^2}{1 + \frac{fV}{\gamma}}$$

We can see from the form of the equation of motion that if ϕ begins from rest on a steep potential, it will quickly accelerate for a while, until $\dot\phi$ approaches the speed limit $\sqrt{\frac{1}{f}} = \frac{\sqrt{\lambda}}{\phi^2}$. As ϕ speeds up, γ increases from 1 to a large number, at which point the last two terms in the equation of motion can be ignored. Near the speed limit, we can see from the form of w that there will be inflation as long as $\frac{fV}{\gamma} \gg 1$.