Maldacena, problem 5

We wish to compute the classical action of unperturbed dS_4 with an S^3 boundary at $\tau = \tau_c$ and the Hartle-Hawking analytic continuation, as a saddle-point approximation for the path integral. The metric is

$$ds^2 = -d\tau^2 + \cosh^2\tau d\Omega_3^2. \tag{1}$$

The action separates into three parts: the Hartle-Hawking continuation, the dS action, and the boundary. First we consider the boundary piece, S_B. The action is just

$$S_B = \frac{R_{dS}^2}{8\pi G_N}\int_{\partial dS_4} K = \frac{R_{dS}^2}{8\pi G_N}\cosh^3\tau_c \int_{S^3} K(\tau_c), \tag{2}$$

where $K = \frac{1}{2}h^{ij}\partial_\tau h_{ij}$ is the (dimensionless) extrinsic curvature of the boundary, h is the metric of the boundary, and R_{dS} is the characteristic radius of the de Sitter space. K is constant on the boundary and equal to

$$\frac{1}{2}\frac{\partial_\tau \cosh^2\tau}{\cosh^2\tau}\delta^{ij}\delta_{ij} = \frac{3}{2}\frac{\sinh(2\tau_c)}{\cosh^2\tau_c}. \tag{3}$$

The integral over S^3 is then just the volume of S^3, which is $2\pi^2$; thus, the boundary action is

$$S_B = \frac{R_{dS}^2}{16\pi G_N}6\pi^2\sinh(2\tau_c)\cosh\tau_c. \tag{4}$$

Next we compute the de Sitter action S_{dS}, between $\tau = 0$ and $\tau = \tau_c$. Again pulling out dimensionful constants, we have

$$S_{dS} = \frac{R_{dS}^2}{16\pi G_N}\int_{dS_4}\sqrt{-g}(R-6), \tag{5}$$

where $3R_{dS}^2$ is the cosmological constant in four dimensions. de Sitter has constant positive curvature, so $(R-6)$ factors out of the integral; in particular, $R = 12$ in four dimensions with R_{dS}^2 factored out. Plugging in the determinant of the metric,

$$\begin{aligned} S_{dS} &= \frac{R_{dS}^2}{16\pi G_N}6\int_0^{\tau_c} d\tau \int d\Omega_3^2\cosh^3\tau \\ &= \frac{R_{dS}^2}{16\pi G_N}36\pi^2\sinh\tau_c\cosh^2\tau_c. \end{aligned} \tag{6}$$

Lastly we look at the Hartle-Hawking continuation. We analytically continue τ onto the imaginary axis: $\tau_E = i\tau$ running from $\frac{\pi}{2}$ to 0. This gives the Euclidean metric

$$ds_E^2 = d\tau_E^2 + \cos^2\tau_E d\Omega_3^2, \tag{7}$$

which is the metric of a 4-sphere. The range $0 \le \tau_E \le \frac{\pi}{2}$ gives a hemispherical "cap" to the bottom of the spacetime, functioning as a boundary in the past. S^4, like de Sitter, is also a space of constant positive curvature $R = 12$, so the action is

$$S_{HH} = \frac{R_{dS}^2}{16\pi G_N} 6 \int_{\frac{\pi}{2}}^{0} d(-i\tau_E) \cos^3 \tau_E \int d\Omega_3^2$$
$$= \frac{R_{dS}^2}{16\pi G_N}(6i)\frac{\int d\Omega_4^2}{2} = \frac{R_{dS}^2}{16\pi G_N}(8\pi^2 i). \tag{8}$$

Thus the total action is

$$S = \frac{R_{dS}^2}{16\pi G_N}\pi^2(8i + 36\sinh\tau_c\cosh^2\tau_c + 6\sinh(2\tau_c)\cosh\tau_c) \tag{9}$$

The wavefunction of the spacetime goes as e^{iS}. The latter two terms of the action give oscillating phase as τ_c increases, but the first term—the one from the Hartle-Hawking continuation—produces an exponential decay with no dependence on the cutoff:

$$\Psi \sim \exp\left[-\frac{R_{dS}^2\pi}{2G_N}\right]. \tag{10}$$

That is, the probability of finding this empty de Sitter universe is proportional to $|\Psi|^2 \sim \exp[-R_{dS}^2\pi/G_N]$, and $R_{dS}^2\pi/G_N = \Gamma$ has the appearance of a tunneling rate. Interestingly, the cutoff-independent behavior of Ψ above is precisely equal to that of Z in problem 1.4, with the AdS radius replaced by the dS radius.

It should be noted that we could just as easily have defined $\tau_E = -i\tau$, or equivalently integrated it from $-\frac{\pi}{2}$ to 0. That would have introduced a minus sign on the cutoff-independent part of the action, and thus taken the exponential behavior of Ψ to be growing rather than decaying. These correspond to two different ways to attach the cap, or two different boundary conditions for the action. Superpositions of these two Ψs solve the Wheeler-de Witt equation, since each of them does separately.

Creminelli, problem 2

We are given:

$$\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\dots\zeta_{\vec{k}_n}\rangle = (2\pi)^3\delta\left(\sum_i \vec{k_i}\right)F(k_i) \tag{11}$$

$\zeta(\vec{x},t)$ appears in an exponential in the line element:

$$ds^2 = -dt^2 + e^{2\zeta(\vec{x},t)}a^2(t)\delta_{ij}dx^i dx^j \tag{12}$$

and hence must be dimensionless. Define the dilation symmetry $\{x_i,\eta\} \to \lambda\{x_i,\eta\}, k_i \to k_i/\lambda$. Then from the Fourier transform:

$$\zeta_{\vec{k}} = \int d^3x\, \zeta(\vec{x},t)e^{i\vec{k}\cdot\vec{x}} \tag{13}$$

we see that $\zeta_{\vec{k}}$ transforms as $\zeta_{\vec{k}} \to \lambda^3\zeta_{\vec{k}}$. Analogously, $\delta\left(\sum_i \vec{k_i}\right) \to \lambda^3\delta\left(\sum_i \vec{k_i}\right)$.

Therefore $\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\dots\zeta_{\vec{k}_n}\rangle$ scales as λ^{3n}, and $F(k_i)$ must scale as λ^{3n-3} for the scalings of both sides of eqn.(11) to balance. This can only be achieved if $F(k_i)$ has the dependence k^{-3n+3}.

Susskind, problem 5

We have three vacua which we call 0, 1, and 2. 0 is terminal. After some small Δt the probability that I find myself in one vacuum or another is given by

$$\left(\begin{array}{ccc} 1 & \gamma_{10} & \gamma_{20} \\ 0 & 1-\gamma_{10}-\gamma_{12} & \gamma_{21} \\ 0 & \gamma_{12} & 1-\gamma_{21}-\gamma_{20} \end{array}\right) \times \left(\begin{array}{c} P_0(t) \\ P_1(t) \\ P_2(t) \end{array}\right) = \left(\begin{array}{c} P_0(t+\Delta t) \\ P_1(t+\Delta t) \\ P_2(t+\Delta t) \end{array}\right) \tag{14}$$

where $\gamma_{12} =$ tunneling rate/voltime $\cdot\, \Delta t$ from $1 \to 2$ and $P_i(0)$ is the probability of finding oneself in that particular vacuum.

We can rearrange into a differential equation:

$$\left(\begin{array}{ccc} 0 & \Gamma_{10} & \Gamma_{20} \\ 0 & 0-\Gamma_{10}-\Gamma_{12} & \Gamma_{21} \\ 0 & \Gamma_{12} & 0-\Gamma_{21}-\Gamma_{20} \end{array}\right) \times \left(\begin{array}{c} P_0(t) \\ P_1(t) \\ P_2(t) \end{array}\right) = \left(\begin{array}{c} \dot{P}_0(t) \\ \dot{P}_1(t) \\ \dot{P}_2(t) \end{array}\right) \tag{15}$$

where $\Gamma_{12} =$ tunneling rate/voltime from $1 \to 2$. This is the transition matrix.

The eigenvalues of this matrix are give by the solutions to

$$\lambda\left[(\lambda+\Gamma_{10}+\Gamma_{12})(\lambda+\Gamma_{21}+\Gamma_{20})-\Gamma_{12}\Gamma_{21}\right]=0. \tag{16}$$

$\lambda = 0$ is a solution (the transition matrix has a zero eigenvalue). By inspection we see that the associated (normalized) eigenvector is simply sitting in the terminal vacuum, that is:

$$\left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right)$$

Writing $\Gamma_{10} + \Gamma_{12} = a > 0$ and $\Gamma_{21} + \Gamma_{20} = b > 0$ and solving for the other eigenvalues we have

$$\lambda = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4ab + 4\Gamma_{12}\Gamma_{21}}}{2} \tag{17}$$

the solutions are real as discriminant can be written as

$$\begin{aligned} (a-b)^2 + 4\Gamma_{12}\Gamma_{21} &= 4\Gamma^2\text{'s} - 2(4\Gamma\Gamma \text{ perms}) + 2(2\Gamma\Gamma \text{ perms}) + 4\Gamma_{12}\Gamma_{21} && (18) \\ &= 4\Gamma^2\text{'s} + 2\Gamma_{12}\Gamma_{21} - 2(3\Gamma\Gamma \text{ perms}) + 2(2\Gamma\Gamma \text{ perms}) && (19) \\ &\leq (a+b)^2 && (20) \end{aligned}$$

as we assume that all Γ's are > 0. Therefore, the square root of the discriminant $< a + B$ and we see that the other two eigenvalues are negative.

We can write our equations in terms of our linearly independent eigenvectors. Some initial state is given by

$$\alpha\,|\lambda_0\rangle + \beta\,|\lambda_1\rangle + \gamma\,|\lambda_2\rangle \tag{21}$$

where the 1st and 2nd eigenvectors contain the 1 and 2 vacua in addition to part of the terminal vacua (generically).

Solving the transition equation we have that

$$|\text{state}(t)\rangle = \alpha\,|\lambda_0\rangle + \beta e^{-|\lambda_1|}\,|\lambda_1\rangle + \gamma e^{-|\lambda_2|}\,|\lambda_2\rangle \tag{22}$$

we can see immediately that the probability of finding oneself in the 1 or 2 vacua decays exponentially. In other words, an observer following these transitions finds themselves more and more likely to be in the terminal vacuum.

However, if we examine a single time step we see that the *number* of these 1 and 2 vacua grow with time provided the decay rates are smaller that the expansion. Use the percolation/lattice model.

Say we have some number, A, of lattice sites occupied by the 1 vacuum, while B occupy 2. We will deal with the tunneling and then deal with the growth. Doing things in this order is most conservative. After one time step the number goes to

$$\begin{aligned} \text{tunneling} &\;\rightarrow\; A(1-\gamma_{12}-\gamma_{10}) + B(\gamma_{21}) && (23)\\ \text{growth} &\;\rightarrow\; g\left[A(1-\gamma_{12}-\gamma_{10}) + B(\gamma_{21})\right] && (24)\\ &\;\geq\; A(1-\gamma_{12}-\gamma_{10}) && (25) \end{aligned}$$

therefore, if $g \cdot (1-\gamma_{12}-\gamma_{10}) > 1$ is a sufficient condition for the number of lattice sites occupied by vacuum 1 to grow every time step. That is, the number of 1 vacua (the same argument of course applies to the 2 vacua) to grow with time.

Silverstein, problem 1

a) If we want to have our CMB observed today, it is required about 60 e-folding during inflation. Given the potential $\mu^{4-p}\lambda\phi^p$, the e-foldings

$$N_e = \int H\mathrm{d}t = \int H\frac{\mathrm{d}\phi}{\dot{\phi}} = \int \frac{3H^2}{-V'}\mathrm{d}\phi = \int \frac{V}{-V'} = -\int_{\phi_i}^{\phi_f}\frac{\phi}{p}\mathrm{d}\phi = \frac{1}{2p}\left(\phi_i^2 - \phi_f^2\right) \tag{26}$$

Because $\phi > M_p$ and ϕ_f is derived from that the slow roll parameter is order one, then we can conclude that $\phi_f^2 \ll \phi_i^2$. Restoring the M_p,

$$N_e = \frac{\phi_i^2}{2pM_p^2} \tag{27}$$

And there is another constraint from COBE

$$\langle\zeta\zeta\rangle = \frac{1}{k^3}\frac{H^4}{2\dot{\phi}^2} = \frac{1}{k^3}\frac{H^2}{2\left(\frac{V'}{V}\right)^2} = \frac{1}{k^3}\frac{\lambda\mu^{4-p}\phi^{p+2}}{6p^2} \sim \frac{1}{k^3}COBE \sim \frac{1}{k^3}10^{-10} \tag{28}$$

According to the above two equation (26) and (28), we obtain the conditions on μ,

$$\mu = \left(\frac{6\times 10^{-10}\times p^2}{\lambda\left(2pN_e\right)^{\frac{p+2}{2}}}\right)^{\frac{1}{4-p}} \tag{29}$$

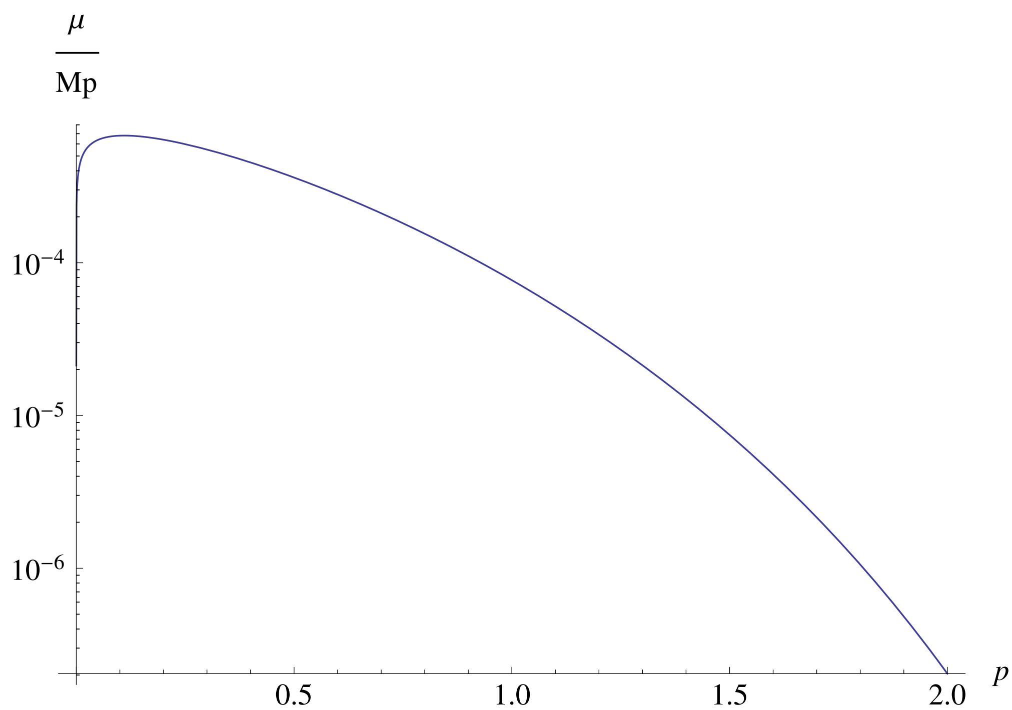


Figure 1: $\mu(p)$

b) We set $\lambda = 1$ to check the scale μ as a function of p which is shown in fig. 1. μ which is much smaller than M_p or Hubble scale will have radiative corrections from loop effect. The easy way to think about is the case $p = 2$, which corresponds to the mass term of inflaton. The loop effect (with some other large mass particles coupling to inflaton) will make the mass the order of the new physics. Assuming the potential is not the complete theory, there is another nonrenormalizable terms. For example, 6 dimension term

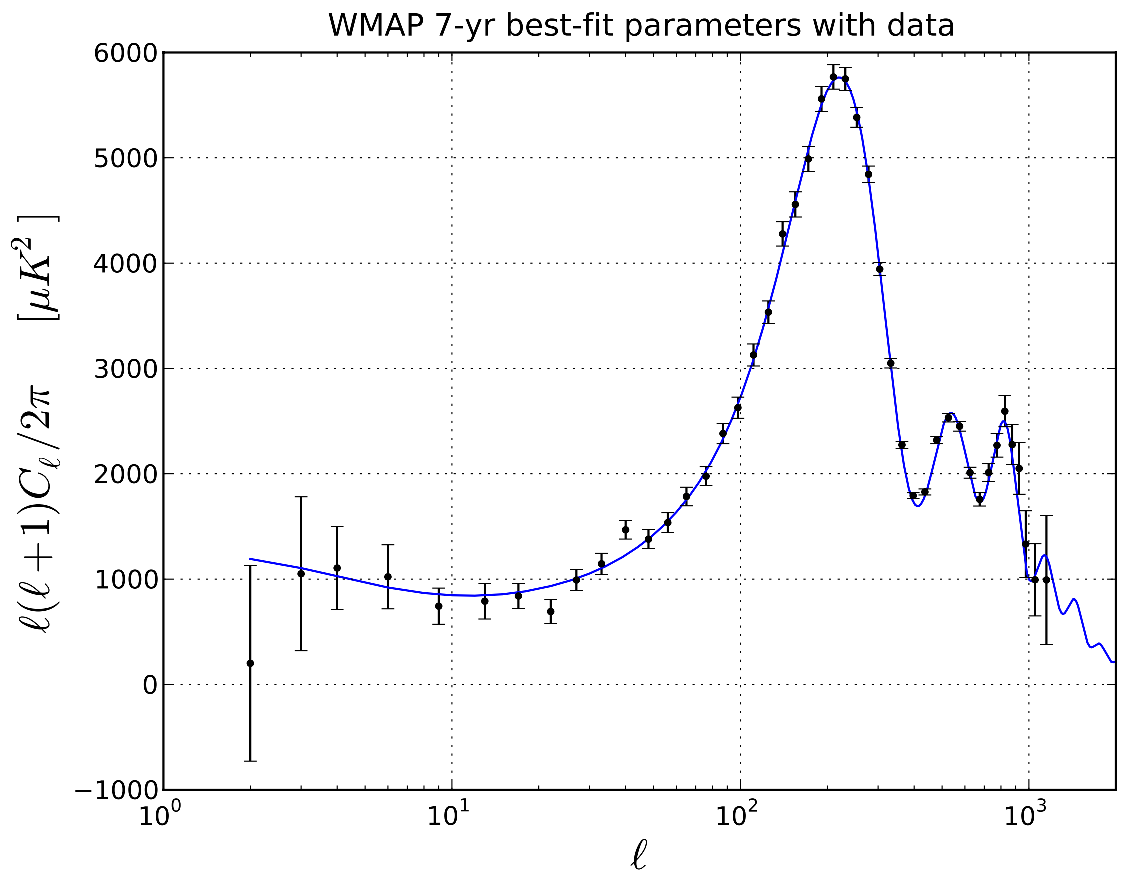
$$L = -\phi^6 \mu^{-2} \tag{30}$$

If $\mu \ll M_p$ and $\phi > M_p$, then these terms will dominate, and there is η problem. However, if there is (approximate) shift symmetry in the theory, there is no other terms correcting the potential in quantum level, then it is stable. The potential is not fine-tuned as it is technically natural, ie. a new symmetry is obtained when $\lambda \to 0$.

c) 1) Need to ensure moduli remain stabilized throughout inflation, as they can disrupt the potential if they start to roll. 2) The potential can also be destabilized if the inflaton rolls too far, eg. $\frac{\Delta\phi}{M_P} \sim 10$ 3) New degrees of freedom might appear as the inflaton moves down a warped throat. The gravitational redshifting can cause fields that are normally heavy in the rest of the compactification to become light there. 4) A general string compactification involves multiple backreaction effects which are important.

Zaldarriaga, Q5 - Solution from Group 5

WMAP best-fit spectrum:



Explanation for the effects of cosmological parameters on the CMB TT power spectrum:

$\Omega_b h^2$

• Increasing $\Omega_b h^2$ causes greater disparity between the heights of odd and even peaks. Eg, note that the $\Omega_b h^2 = 0.03$ curve is highest in the first peak but lowest in the second. This can be understood by analogy with a driven SHO, for which the equation of motion is:

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m} \tag{1}$$

where F_0 is a forcing constant. The general solution (having applied boundary conditions to eliminate a sine solution) is:

$$x = A\cos(\omega t) + \frac{F_0}{m\omega^2} \tag{2}$$

Increasing the baryon density makes the photon-baryon fluid heavier, which decreases its frequency of oscillation. Using eq.(2), the offset of the motion from zero then increases. Modes are not oscillating about their mean, but some positive value. When oscillations are squared to form the power spectrum, this leads to alternating heights between the odd and even peaks. Conversely, lower $\Omega_b h^2$ → higher

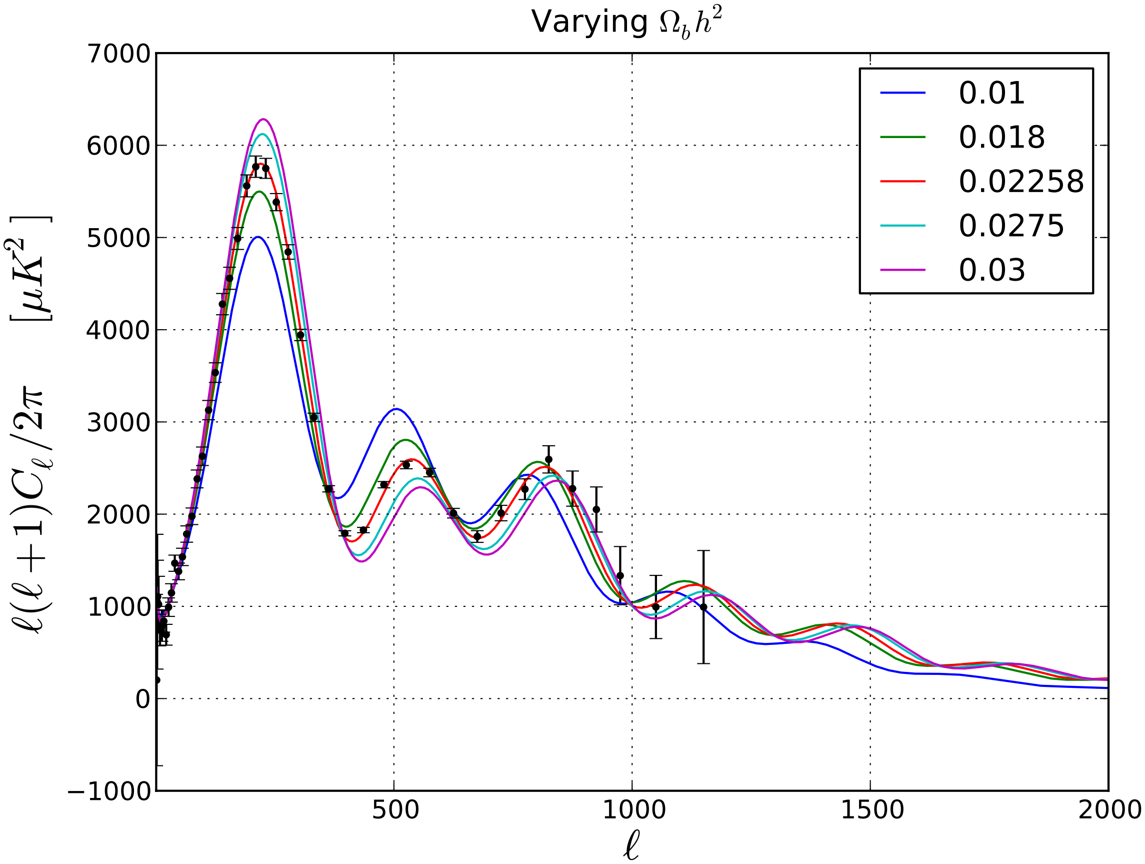


Figure 1: The effects of varying $\Omega_b h^2$. $\Omega_b h^2 = 0.02258$ is the WMAP best-fit value.

$\omega \to$ less offset $\to$ less disparity in peak heights.

• An alternative way of explaining the same idea: the odd peaks correspond to overdensities lining up with gravitational wells at recombination, whilst even peaks correspond to underdensities—remember both contribute equally to a power spectrum. Increasing $\Omega_b h^2$ deepens potential wells, increasing overdensities. It is harder for photons to climb out of these deep wells, so the magnitude of underdensities is reduced.

• Peak spacing—this increases for increasing baryon density. From eq.(2) above, peaks occur at $t = n\pi/\omega$. t corresponds to the interval between the time a mode enters the horizon and recombination, which decreases for smaller k (or equivalently l). Small k are larger modes, which enter later. So the spacing of peaks in t corresponds to a spacing in k between modes which have reached maxima/minima of their oscillations at the time of last scattering. Increase $\Omega_b h^2 \to$ decrease $\omega \to$ increase spacing between peaks.

• The damping tail begins in at higher l-values for larger baryon densities. The mean free path of a photon during recombination is very approximately given by

$$\lambda \approx \frac{1}{\sqrt{n_e \sigma_T H}} \tag{3}$$

During the ionized era $n_e \propto \Omega_b h^2$. So for larger baryon densities the photon mean free path is shorter, and hence the characteristic scale on which perturbations get washed out is smaller. This means the damping is less pronounced—effectively it kicks in at higher l-values.

<u>$\Omega_m h^2$</u>

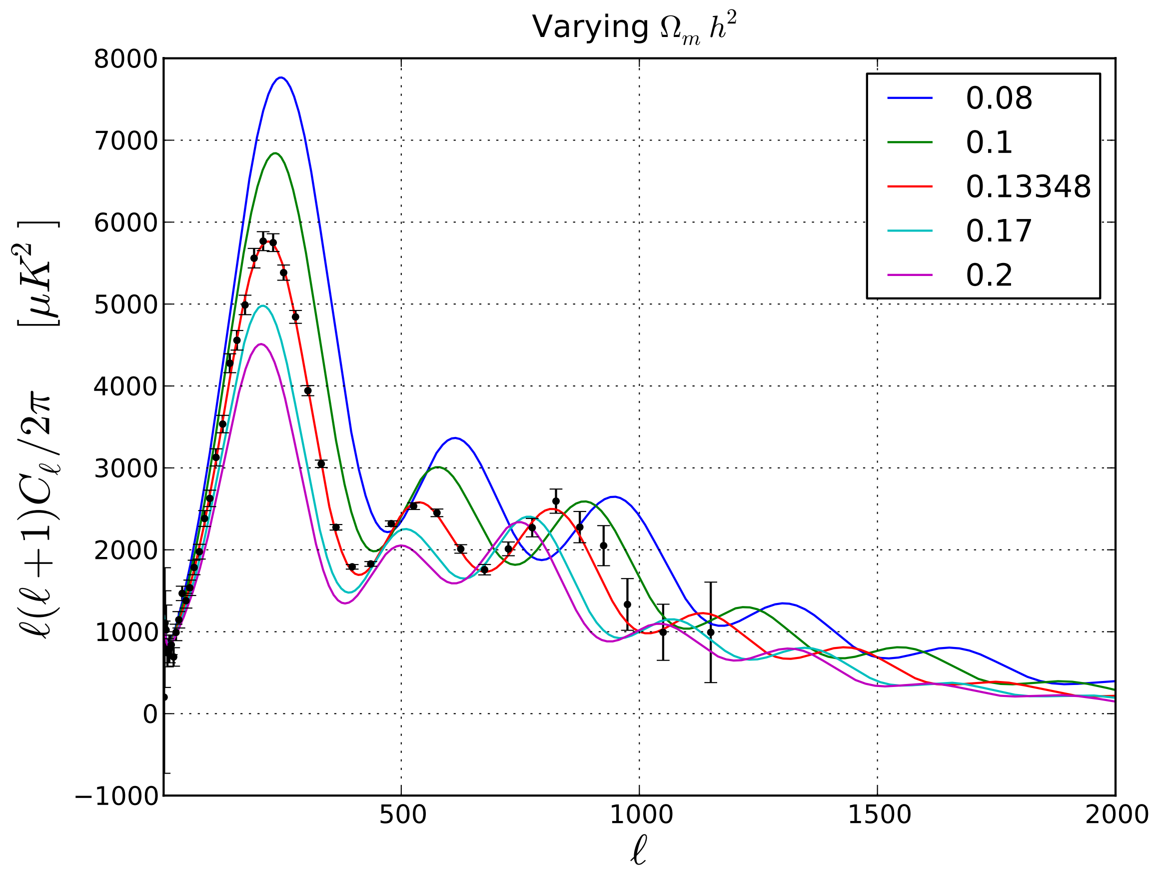


Figure 2: The effects of varying $\Omega_m h^2$. $\Omega_m h^2 = 0.13348$ is the WMAP best-fit value.

• Changing $\Omega_m h^2$ changes the heights of the peaks through an early ISW effect. Potentials decay during the radiation era but are constant during the matter era (as can be seen from the time-dependence of the Poisson equation). If $\Omega_m h^2$ is low, matter-radiation equality occurs closer to recombination. The decay of potentials from the radiation era then has to be taken into account at recombination (of course recombination is still happening in the matter era, but the transition from radiation to matter domination isn't instantaneous, so it takes a while for the potentials to stabilize).

The ISW contribution to the power spectrum is proportional to $\left(\dot{\Phi} + \dot{\Psi}\right)^2$, so changing potentials give a boost in power over static potentials. Hence peak amplitudes increase as $\Omega_m h^2$ decreases.

• Increasing $\Omega_m h^2$ shifts the location of the first peak to lower l-values. Because we're keeping Ω_Λ fixed, Ω_m is fixed and hence we must be increasing H. If we consider the age of the universe to be given by H^{-1} (to first order!) this corresponds to a younger universe. The surface of last scattering would then be closer to us, so peaks subtend a larger angle and hence occur at lower l.

$\underline{\Omega_\Lambda}$

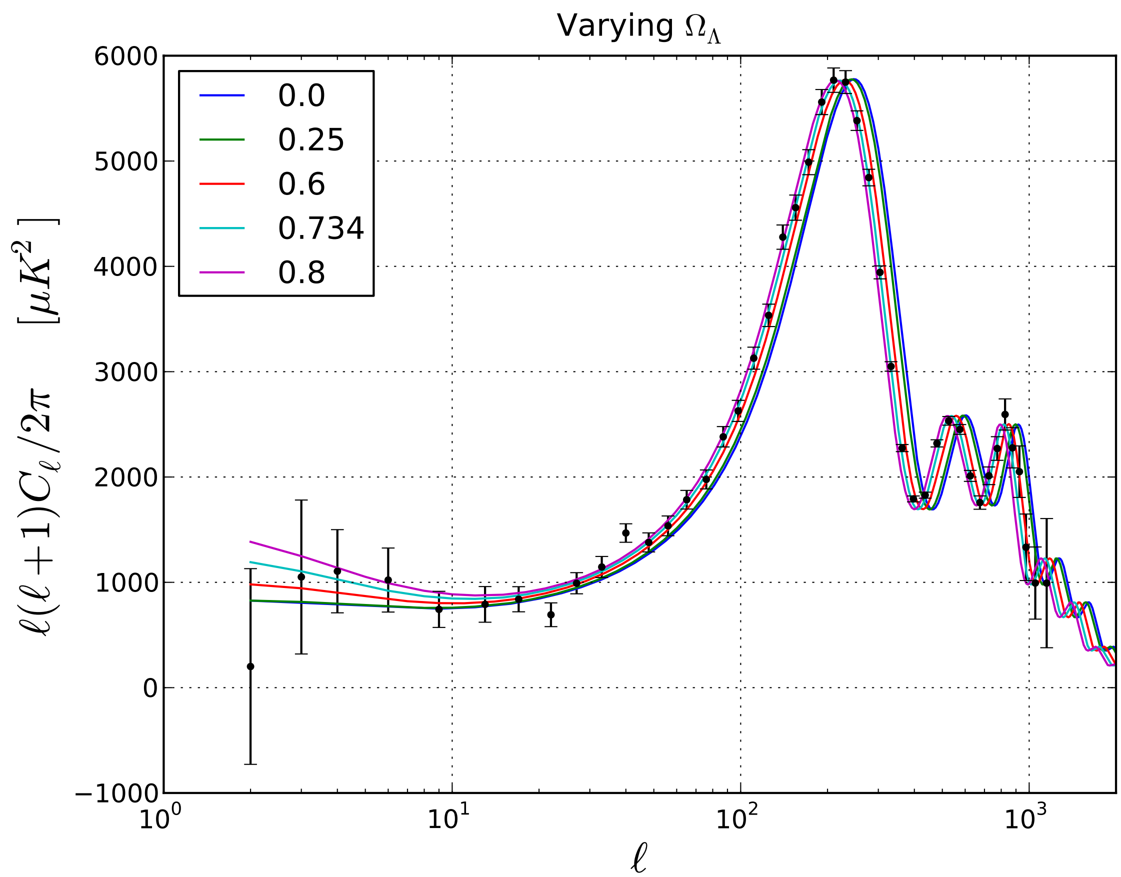


Figure 3: The effects of varying Ω_Λ. $\Omega_\Lambda = 0.734$ is the WMAP best-fit value. NB: WMAP value is shown in a different colour to previous plots.

• Ω_Λ effects the large-angle CMB through the late-time ISW effect. The dependence is exactly the same as given above, except now the cause of the potential decay is the late-time acceleration of the Λ-dominated era. Formally the contribution is integrated over all times since last scattering, but the integral is dominated by recent times. Recent effects correspond to large angles on the sky, hence low l-values are affected.

• The peak positions also shift to smaller l if Ω_Λ is larger. This is due to a change in conformal time intervals in a Λ-dominated universe. To explain: if $\Omega_b h^2$ is held fixed, then the sound horizon at last scattering is fixed. The angle this fixed physical scale occupies on the sky is roughly:

$$\theta_* = \frac{\text{fixed scale}}{\text{radial distance}} \approx \frac{k_*^{-1}}{\eta_0 - \eta_{rec}} \tag{4}$$

Consider a short conformal time interval:

$$d\eta = \frac{dt}{a} = \frac{da}{a^2 H} = \frac{da}{a^2\sqrt{\frac{1}{3M_P^2}\rho_M + \frac{\Lambda}{3}}} \tag{5}$$

In a universe with $\Lambda \neq 0$ the interval $d\eta$ is shorter than in one with $\Lambda = 0$. Therefore the denominator of eq.(4) is smaller and the peaks shift to larger angular scales $\rightarrow$ lower l-values.

Error Estimates

We will continue to consider varying one parameter at a time - not ideal for a degenerate set of parameters!

For the $\Omega_b h^2$ graph, the error bars around the first peak look to be roughly ± 100. The WMAP (red) curve sits roughly in the middle of the $\Omega_b h^2 = 0.018, 0.0275$ curves, which have a difference of about 600 in height around the first peak. So we can estimate the error on $\Omega_b h^2$ to be approximately $100/600 \times (0.0275 - 0.018) \approx 0.016$. This is about 3 times larger than the real WMAP error bars of $\sim \pm 0.0006$. (After all, we are just eyeballing it...)

Applying a similar approach to the $\Omega_m h^2$ graph (again looking around the first peak), I estimate $100/1800 \times (0.17 - 0.1) \approx 0.004$. This is a better match to the WMAP quoted errors of ± 0.0055.

The error on Ω_Λ is somewhat harder to estimate, as varying Λ doesn't cause a change in peak heights. Looking at the RHS of the first peak, it's clear that $\Omega_\Lambda = 0.6, 0.8$ are ruled out. I would say that the gap between the WMAP and $\Omega_\Lambda = 0.6$ curves would need to be about one third it's size to achieve consistency with the error bars. Hence a very rough estimate of the error would be $1/3 \times (0.734 - 0.6) \approx 0.045$. The WMAP value is ± 0.029, so we're not too far off.