Group 7

Problem Set Solutions

Maldacena #1.

a) The action is:

$$S = \int d^4x \sqrt{-g}\,\left(\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi\right) \quad \text{and} \quad ds^2 = \frac{-d\eta^2 + d\vec{x}^2}{\eta^2} \quad \Rightarrow \sqrt{-g} = \frac{1}{\eta^4}$$

$\Rightarrow$ the EoM is: $\nabla_\mu \partial^\mu \phi = 0$

$$\Rightarrow \eta^4 \frac{\partial}{\partial\eta}\left(-\eta^2 \frac{1}{\eta^4}\dot\phi\right) + \eta^4 \frac{\partial}{\partial x}\left(\eta^2 \frac{1}{\eta^4}\nabla^2\phi\right) = 0 \quad \left(\text{using } \nabla_\mu V^\mu = \frac{1}{\sqrt{g}}\frac{\partial(V^\mu\sqrt{g})}{\partial x^\mu}\right)$$

$$\eta^4(-1)\frac{(-2)}{\eta^3}\dot\phi - \eta^2\ddot\phi + \eta^2\nabla^2\phi = 0$$

$$\ddot\phi - \frac{2}{\eta}\dot\phi - \nabla^2\phi = 0 \quad \Rightarrow \text{to Fourier space:}$$

$$\ddot\phi_k - \frac{2}{\eta}\dot\phi_k - k^2\phi_k = 0$$

If we now expand each $\phi_k = a_k^+ f_k + a_k f_k^*$ (with f_k & f_k^* are the positive and negative frequency mode solution) $\Rightarrow$ the f_k EoM ~~are~~:

$$\boxed{\ddot f_k - \frac{2}{\eta}\dot f_k - k^2 f_k = 0}$$

To solve this equation, change variable: $z = k\eta$

$$\Rightarrow \ddot z f \quad 2z\dot f + z^2 f = 0$$

This can be rewritten as a confluent hypergeometric equation. A solution is:

$$f(\eta) = e^{ik\eta}\,\Phi(-1,-2;-2ik\eta)$$

with Φ the hypegeometric confluent function it can be related to Bessel functions $J_{3/2}$ as:

$$\Phi(-1,-2;-2ik\eta) = \Gamma(-\tfrac{1}{2})\,e^{-ik\eta}\left(-\tfrac{1}{2}k\eta\right)^{3/2} J_{-\frac{3}{2}}(-k\eta)$$

But the Bessel functions $J_{3/2}$, $J_{-3/2}$ have the simple form:

$$J_{3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{-3/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\left(-\sin x - \frac{\cos x}{x}\right)$$

So we can rewrite $f(\eta)$ using $x = -z = k\eta$:

$$f(\eta) = A_1\,\Gamma(-\tfrac{1}{2})\left(-\tfrac{1}{2}\right)^{3/2}(-z)^{3/2}\left(\tfrac{2}{\pi}\right)^{1/2}(-z)^{-1/2}\left(\frac{\sin -z}{(-z)} - \cos(-z)\right) \quad \text{with } A_1 = \text{cst}$$

$$= A_1(-z)\left(-\sin(-z) - \frac{\cos z}{(-z)}\right) \quad \text{(redefining the constant } A_1\text{)}$$

$$f_1(\eta) = A_1\,(-k\eta)\sin(-k\eta) + \cos(-k\eta)$$

And a linearly independant solution would be:

$$f_2(\eta) = A_2\sin(-k\eta) - (-k\eta)\cos(k\eta)$$

We then find linear combinaisons of f_1 & f_2 s.t we get the left and right (positive and negative) mode solutions we are looking for, with prop to $e^{-ik\eta}$ & $e^{+ik\eta}$ respectively. We find:

$$f \sim [1 - i(-k\eta)]\,e^{-ik\eta} \quad ; \quad f^* \sim [1 + i(-k\eta)]\,e^{ik\eta} \quad \text{up to an overall norm.}$$

b) so we have expanded each ϕ_k mode as:

$$\phi_k = A\, a^+ (1+ik\eta) e^{-ik\eta} + a(1-ik\eta) e^{ik\eta}$$

the conjugate momentum therefore reads:

$$\pi_k = \frac{\partial L}{\partial(\partial_\eta \phi_k)} = \frac{1}{\eta^4} g^{\mu\nu} \frac{\partial\left(-\frac{1}{2}\partial_\mu\phi\,\partial_\nu\phi\right)}{\partial(\partial_\eta\phi_k)} = \frac{-1}{\eta^2}\partial_\eta\phi$$

$$= \frac{-1}{\eta^2} A\left(a^+[(\cancel{1}+ik\eta)(-ik) + (\cancel{ik})]e^{-ik\eta} + a[(\cancel{1}-ik\eta)ik + (\cancel{-ik})]e^{ik\eta}\right)$$

$$= \frac{-A}{\eta^2}\left[k^2\eta a^+ e^{-ik\eta} + k^2\eta a e^{ik\eta}\right]$$

$$= \frac{-A}{\eta} k^2 (a^+ e^{-ik\eta} + a e^{ik\eta})$$

We now impose $[\phi,\pi] = i$ & $[a,a^+]=1$ $\quad aa^+ - a^+a = 1 \Rightarrow a^+a = aa^+ - 1$

$$\Rightarrow [\phi,\pi] = -A^2\Big(\left(a^+(1+ik\eta)e^{-ik\eta} + a(1-ik\eta)e^{ik\eta}\right)\left(\frac{k^2}{\eta}\right)(a^+e^{-ik\eta} + ae^{ik\eta}) - \frac{k^2}{\eta}(a^+e^{-ik\eta} + ae^{ik\eta})$$
$$\times\left(a^+(1+ik\eta)e^{-ik\eta} + a(1-ik\eta)e^{ik\eta}\right)\Big) = i$$

$$i = -A^2\frac{k^2}{\eta}\left(a^+a(\cancel{1}+ik\eta) + aa^+(\cancel{1}-ik\eta) - a^+a(\cancel{1}-ik\eta) - aa^+(\cancel{1}+ik\eta)\right)$$

$$i = \frac{-A^2k^2}{\eta}\left((\cancel{aa^+}-1)ik\eta + aa^+(\cancel{-ik\eta}) + (-\cancel{aa^+}+1)(-ik\eta) - aa^+\cancel{ik\eta}\right)$$

$$\cancel{i} = \frac{-A^2k^2}{\cancel{\eta}}(-2\cancel{i}k\cancel{\eta}) \qquad \Rightarrow \boxed{A = \frac{1}{\sqrt{2k^3}}}$$

c) define $a_k|BD\rangle = 0$. This is a reasonable definition, because at early enough time when a_k, a_k^+ are the annihilation and creation operators of the k-mode vaccum state, the physical wavelength corresponding to k is very small and well inside H^{-1} the Hubble radius, and so the (physical) scale k feels as if in Minkowski space. it is therefore natural to choose the Bunch-Davis vaccum, ie the Minkowski vacuum of a comoving observer in the far past $\tau \to \infty$ (when all k-modes are inside)

$$\Rightarrow \langle BD|\, \phi_{\vec{k}}(\eta)\,\phi_{(\vec{k})}(\eta')\,|BD\rangle = \frac{1}{2k^3}\langle BD|(1-ik\eta)e^{ik\eta}a_k(1+ik\eta')e^{-ik\eta'}a_k^+|BD\rangle$$

$$= \frac{1}{2k^3}(1-ik\eta)(1+ik\eta')e^{ik(\eta-\eta')}$$

$$= \frac{1}{2k^3}(1+k^2\eta^2)$$

d) Fourier transform the result from c) to position space:

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} (1+k^2\eta^2) e^{i\vec{k}\cdot\vec{x}} = \int \frac{dk\,d\theta\,d\phi}{(2\pi)^3} \frac{k^2 \sin\theta}{(2k)^3} (1+k^2\eta^2) e^{ikx\cos\theta}$$

$$= \frac{2\pi}{(2\pi)^3} \frac{1}{2} \int dk \left[\int d\theta \sin\theta\, e^{ikx\theta}\right] \frac{1}{k} (1+k^2\eta^2)$$

$$= \frac{1}{2(2\pi)^2} \int dk \left(-\frac{1}{ikx}\right) e^{ikx\cos\theta}\Big|_0^\pi \frac{1}{k} (1+k^2\eta^2)$$

$$= \frac{1}{(2\pi)^2} \int dk \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right)\left(\frac{1+k^2\eta^2}{k^2 x}\right)$$

$$= \frac{1}{4\pi^2} \left[\int dk \frac{\sin(kx)}{k^2 x} + \eta^2 \int dk \frac{\sin kx}{x}\right] \leftarrow$$

$$= \frac{1}{4\pi^2} \left[-\frac{\sin xk}{x(1)k}\Big|_0^\infty + \frac{x}{\cancel{x}} \int \frac{\cos kx}{k} dk + \eta^2 \int dk \frac{\sin kx}{x}\right]$$

$$= \frac{1}{4\pi^2} \left[-\frac{\sin(xk)}{kx}\Big|_0^\infty + \ln|kx|\Big|_0^\infty + \sum_{n=1}^{\infty} (-1)^n \frac{(kx)^{2n}}{2n\cdot(2n)!}\Big|_0^\infty + \underbrace{\eta^2 \left(-\frac{\cos(kx)}{x^2}\right)\Big|_0^\infty}_{\to 0 \text{ as } \eta \to 0}\right]$$

$$= \frac{1}{4\pi^2} \left[+\underbrace{\frac{\sin(\varepsilon x)}{\varepsilon x}}_{\to 1} + \cancel{\ln|kx|}\Big|_{k=\infty} - \ln\left|\frac{1}{L} x\right| \cancel{+ \ln|kx|}\Big|_{k=\infty} - \gamma_E \right.$$
$$\left. - \ln\left|\frac{x-x'}{L}\right| \right]$$

$$= \frac{1}{4\pi^2} \left[1 - \ln\left|\frac{x-x'}{L}\right|\right] - \gamma_E \Big]$$

6. (Creminelli)

Calculate $\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle_{1\eta}$

For simplicity we set $H = 1$.

$$\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle = \langle 0 | U^{-1} \phi_{k_1} \phi_{k_2} \phi_{k_3} U | 0 \rangle$$

with $U = T e^{-i \int_{t_0}^{t} H_{int}(t') dt'}$, $|t_0| \ll 1$

$$\overset{\text{1st order}}{\simeq} 1 - i \int_{t_0}^{t} H_{int}(t') dt'$$

$$= \underbrace{\langle 0 | \phi_{k_1} \phi_{k_2} \phi_{k_3} | 0 \rangle}_{=0} - i \int_{t_0}^{t} dt' \langle 0 | [\phi_{k_1} \phi_{k_2} \phi_{k_3} , \underbrace{H_{int}(t')}_{= \int d^3x \sqrt{-g} \tilde{H}_{int}(t',x)}] | 0 \rangle$$

$$\tilde{H}_{int}(t',x) = -\frac{\mu}{6} \phi^3(t',x)$$

$$= \frac{i\mu}{6} \int_{t_0}^{t} dt' d^3x \sqrt{-g} \; \langle 0 | [\phi_{k_1} \phi_{k_2} \phi_{k_3} , \phi^3(t',x)] | 0 \rangle$$

Now $\phi(t',x) = \int \frac{d^3k}{(2\pi)^3} \phi_k(t') e^{-i\vec{k}\cdot\vec{x}}$

$$= \frac{iM}{6}\int_{t_0}^{t} \underbrace{dt' d^3x \sqrt{-g}}_{d\eta' d^3x \frac{1}{\eta'^4}} \int \frac{d^3k_4}{(2\pi)^3}\frac{d^3k_5}{(2\pi)^3}\frac{d^3k_6}{(2\pi)^3} e^{-i(\vec{k}_4+\vec{k}_5+\vec{k}_6)\cdot x}$$

$$\times \langle 0|[\phi_{k_1}(\eta)\phi_{k_2}(\eta)\phi_{k_3}(\eta), \phi_{k_4}(\eta')\phi_{k_5}(\eta')\phi_{k_6}(\eta')]|0\rangle$$

$$= \frac{iM}{6}(2\pi)^3\delta^{(3)}(\vec{k}_4+\vec{k}_5+\vec{k}_6)\int_{\eta_0}^{\eta}\frac{d\eta'}{\eta'^4}\frac{d^3k_4}{(2\pi)^3}\frac{d^3k_5}{(2\pi)^3}\frac{d^3k_6}{(2\pi)^3}$$

$$\times \underbrace{\langle 0|[\phi_{k_1}(\eta)\phi_{k_2}(\eta)\phi_{k_3}(\eta), \phi_{k_4}(\eta')\phi_{k_5}(\eta')\phi_{k_6}(\eta')]|0\rangle}_{\equiv A}$$

To evaluate A use Wick's theorem and keep only terms which contract $\{k_1, k_2, k_3\}$ with $\{k_4, k_5, k_6\}$, for instance $\phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}\phi_{k_5}\phi_{k_6}$, since only those will give an overall factor $\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)$. These are 6 contractions for each sammand in A. After integration over $d^3k_4\, d^3k_5\, d^3k_6$ these 6 terms turn out to be the same. Also the second sammand in A is the complex conjugate of the first one

$$= \frac{iM}{\cancel{6}}(2\pi)^3\,\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\,\cancel{6}\int_{\eta_0}^{\eta}\frac{d\eta'}{\eta'^4}\Big(f^+_{k_1}(\eta)f_{k_1}(\eta')f^+_{k_2}(\eta)f_{k_2}(\eta')f^+_{k_3}(\eta)f_{k_3}(\eta')$$
$$- c.c.\Big)$$

where we have used

$$\phi_{\vec{k}}(\eta) = a^+_{\vec{k}}f_k(\eta) + a_{-\vec{k}}f^+_k(\eta)\,,\quad f_k(\eta) = \frac{1}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}$$
$$[a_{\vec{k}_1}, a^+_{\vec{k}_2}] = (2\pi)\delta^3(\vec{k}_1+\vec{k}_2)$$

and
$$\phi_{k_1}(\eta)\phi_{k_2}(\eta') = \langle 0|[\phi_{k_1}(\eta),\phi_{k_2}(\eta')]|0\rangle$$
$$= f^+_{k_1}(\eta)f_{k_2}(\eta')(2\pi)^3\delta^{(3)}(\vec{k}_1-\vec{k}_2)$$

Writing $f_{k_1}(\eta')f_{k_2}(\eta')f_{k_3}(\eta') \equiv B(\eta')$ we can write the integral as

$$\int_{\eta_0}^{\eta} \frac{d\eta'}{\eta'^4} \Big(\underbrace{B^+(\eta)\, B(\eta')}_{①} - \underbrace{B^+(\eta')\, B(\eta)}_{②} \Big)$$

Since $B(\eta') \sim e^{-i(k_1+k_2+k_3)\eta'}$ we have to choose $\eta_0 \to -\infty(1+i\epsilon)$ for ① and $\eta_0 \to -\infty(1-i\epsilon)$ for ②.

Effectively, we can write

$$\int_{-\infty}^{\eta} \frac{d\eta'}{\eta'^4} \left(B^+(\eta)\, B(\eta') - B(\eta)\, B^+(\eta') \right) e^{k\epsilon\eta'} \quad , k = k_1+k_2+k_3$$

$$= 2i \int_{-\infty}^{\eta} \frac{d\eta'}{\eta'^4} \, \mathrm{Im}\left[B^+(\eta)\, B(\eta') \right] e^{k\epsilon\eta'}$$

$$= B^+(\eta) \int_{-\infty}^{\eta} \frac{d\eta'}{\eta'^4} \, B(\eta') e^{k\epsilon\eta'} - c.c. \qquad (*)$$

We have $B(\eta') = \dfrac{1}{\sqrt{8k_1^3 k_2^3 k_3^3}} (1+ik_1\eta')(1+ik_2\eta')(1+ik_3\eta') e^{-ik\eta'}$

So the integral $(*)$ contains 4-terms

$$\int_{-\infty}^{\eta} d\eta' \; \eta'^{-n} e^{-ik\eta' + k\epsilon\eta'} \qquad \text{for } n = 1,2,3,4$$

The integrals for $n = 2, 3, 4$ can be reduced to an $n=1$ integral + boundary terms by partial integration.

We want to evaluate $\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle$ at late times, i.e. $\eta \to 0$. Then $B^+(\eta) = B(\eta) = \frac{1}{\sqrt{8 k_1^3 k_2^3 k_3^3}}$.

We obtain (using Mathematica):

$$\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle = -\mu \, (2\pi)^3 \, \delta^{(3)} (\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \frac{1}{12 k_1^3 k_2^3 k_3^3}$$

$$\times \Big[(k_1^2 + k_2^2 + k_3^2)(k_1 + k_2 + k_3) - k_1 k_2 k_3 - (k_1^3 + k_2^3 + k_3^3) \underbrace{\lim_{\eta \to 0} \mathrm{Re} \Big[\int_{-\infty}^{\eta} d\eta' \frac{e^{-ik\eta' + k\epsilon\eta'}}{\eta'} \Big]}_{\text{Note that this part diverges for } \eta \to 0} \Big]$$

$$\int_{-\infty}^{\eta} d\eta' \frac{e^{-ik\eta' + k\epsilon\eta'}}{\eta'} \underset{\eta \to 0}{\simeq} \underset{\uparrow \text{ Euler-Number}}{\gamma} + \mathrm{Log}(k\eta) - i\frac{\pi}{2}$$

So the divergence is logarithmically.

Thus, our final result is

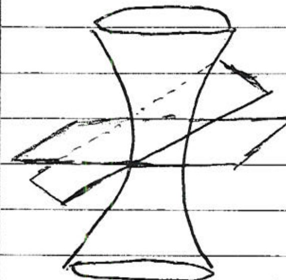
$$\boxed{\begin{aligned}\langle \phi_{k_1} \phi_{k_2} \phi_{k_3} \rangle_{\eta=0} &= \mu \, (2\pi)^3 \, \delta^{(3)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3) \, \frac{1}{12 k_1^3 k_2^3 k_3^3} \\ &\times \Big[k_1 k_2 k_3 - (k_1^2 + k_2^2 + k_3^2)(k_1 + k_2 + k_3) \\ &+ (k_1^3 + k_2^3 + k_3^3) \big(\gamma + \lim_{\eta \to 0} \log(k|\eta|) \big) \Big]\end{aligned}}$$

We can regulate the log-divergence by introducing an infrared cut off.

Creminelli

n° 7. Fixing a gauge and choosing ξ as the dynamical variable for fluctuations picks a prefered time slicing. It therefore breaks time diffeomorphisms and so the special conformal symmetries of the de Sitter group $SO(4,1)$ are broken by the ξ solution.
In the limit where the de Sitter radius goes to ∞, the special conformal symmetries go to the ~~boost~~symmetries, and so in that limit for large R the ξ solution breaks the boosts, which is in agreement with intuition.

1. (Susskind)



ellipsoid

These planes intersect the hyperboloid in positive curvature surfaces. Vertical planes, and slightly tilted away from vertical planes, intersect in negative curvature hyperboloids. As you tilt away from horizontal or vertical and towards lightlike, the curvature decreases. The crossover point is at 45°, which defines a lightlike plane. Putting the flat metric on this surface, we know our total metric is of the form

$$-h(t)^2 dt^2 + f(t)^2 dx_i^2$$

where we have eliminated cross-terms between time and space by picking time orthogonal to the lightlike planes. Rescaling time, we can write this as

$$-d\tau^2 + F(\tau)^2 dx_i^2 \qquad (\text{used } h(t)\,dt = d\tau)$$

Using $R = 12H^2$, which is Einstein's equation for dS, we get a differential equation for $F(\tau)$:

$$\frac{6\left(F'(\tau)^2 + F(\tau)F''(\tau)\right)}{F(\tau)^2} = 12H^2$$

$$\Longrightarrow F(\tau)^2 = C_2 \cosh(2H\tau + C_1)$$

Fixing $F(0) = 1$ and $\frac{(F^2)'}{F^2} = 2H \Longleftrightarrow \frac{F'}{F} = H$

gives $F(\tau) = e^{H\tau}$.

4 (Silverstein)

In an unwarped compact space the brane tension of a cosmic string (D1-brane) is

$$T_1 = \frac{1}{2\pi g_s \alpha'}$$

Unless the volume of the compactification V_6 is larger than a bound

$$V_6 \gtrsim 2 \cdot 10^5 \, g_s \, (2\pi l_s)^6 \qquad \text{(KKLMMT)}$$

the amount of cosmic strings produced at the end of inflation is too high to be in agreement with the current non-observation of cosmic strings. (Cosmic strings emerge by the breaking of an U(1) symmetry by a tachyonic complex scalar that emerges as the brane and anti-brane come closer than $\sim l_s$).

In a waped geometry as is used in $D3\,\overline{D3}$-inflation one does not have to worry about an overproduction for the following reason: The branes annihilate at the tip of the throat. All objects are redshifted down the throat, so

$$T_1 \to \frac{1}{2\pi g_s \alpha'}\, e^{-\frac{4\pi k}{3 g_s M}} \qquad k, M \to \text{flux constants} \in \mathbb{Z}$$

So T_1 is naturally exponentially small and hence cosmic string perturbations to the metric are exponentially suppressed.