PiTP problem set, week 1

Group 8

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1 Problems for dS/CFT (Maldacena)

Problem 2

Recall that a scalar field in de Sitter space can be written as $\phi_k = a^\dagger f + a f^*$, where $f = (2k^3)^{-1/2}(1 + i|k|\eta)e^{-i|k|\eta}$ is a classical solution at a fixed Fourier mode corresponding to the Bunch-Davies vacuum.

As we checked in Problem 1, the momentum-space propagator is

$$\langle \phi_k(\eta)\phi_{k'}(\eta')\rangle = (2\pi)^3\delta^3(k + k')f_k(\eta)f^*_{k'}(\eta'). \tag{1}$$

To compute the three-point function in the interacting vacuum $|\Omega\rangle$, we use the in-in formalism to compute in terms of the free-field Bunch-Davies vacuum $|0\rangle$,

$$\left\langle \Omega|\phi^3(\eta)|\Omega\right\rangle = \left\langle 0|\bar{T}e^{i\int_{-\infty}^{\eta} H_I(\eta')d\eta'}\phi^3(\eta)Te^{-i\int_{-\infty}^{\eta} H_I(\eta')d\eta'}|0\right\rangle, \tag{2}$$

where $H_I(\eta') = \frac{\lambda}{3!}\phi(\eta')^3$.

Expanding the exponentials to first order this is

$$\left\langle \phi^3(\eta)\right\rangle = -\frac{i}{3!}\int_{-\infty}^{\eta} d\eta' \left\langle 0\right| \left[\phi^3(\eta), H_I(\eta')\right] \left|0\right\rangle. \tag{3}$$

We can evaluate this simply using Wick's theorem, as described in Appendix A of Weinberg's paper, hep-th/0506236. For the first order term we connect three external legs at a vertex, and have both a "right" and "left" vertex from the time-ordered and anti-time-ordered products. By the Feynman rules each of these terms gives a product of three propagators. There are six possible permutations, giving an overall factor of 6 that cancels the 3!. The final result in momentum space is

$$\langle \phi_{k_1}(\eta)\phi_{k_2}(\eta)\phi_{k_3}(\eta)\rangle = (2\pi)^3\delta^3(k_1 + k_2 + k_3)\times \tag{4}$$

$$\left[i\lambda\int_{-\infty-i\epsilon}^{\eta} \frac{d\eta'}{\eta'^4} f_{k_1}(\eta)f^*_{k_1}(\eta')f_{k_2}(\eta)f^*_{k_2}(\eta')f_{k_3}(\eta)f^*_{k_3}(\eta') + c.c.\right], \tag{5}$$

where we have made the contour slightly imaginary to pick out the appropriate vacuum. We can evaluate this integral by redefining $\eta' \to \eta' + i\epsilon\eta'$. The following term can be found using Mathematica,

$$\int_{-\infty}^{\eta} d\eta' \frac{1}{\eta'(1+i\epsilon)} e^{-ik\eta'(1+i\epsilon)} = i\left[i\Gamma(ik\eta) + \log(-\eta)\right]. \tag{6}$$

The other terms with higher powers of η' in the denominator can be evaluated integrating by parts and again applying this integral. The final answer has complicated k dependence, and we will not write the full expression here, but it contains a logarithm and divergent powers up through $1/\eta^3$.

Note that we also could have obtained 4 by brute force, by operating the raising and lowering operators inside ϕ on the vaccum. It is straightforward but tedious to check that this gives the same answer.

2 Exercises on Inflation (Creminelli)

Problem 5

The dynamics of the goldstone boson, like axion with decay constant f, is governed by the following action

$$S = \int d^4x\sqrt{-g}\left((\partial_\mu\pi)^2 + \frac{(\partial_\mu\pi)^4}{f^4}\right). \tag{7}$$

On the de Sitter background $ds^2 = 1/(H\eta)(-d\eta^2 + dx^2)$ it reads as

$$S = \int d^4x \frac{1}{H^2\eta^2}\left((\partial_\mu\pi)^2 + H^2\eta^2\frac{(\partial_\mu\pi)^4}{f^4}\right), \tag{8}$$

where indices are contracted by the flat metric. The deviation from gaussianity can be estimated as

$$\frac{\mathcal{L}_4}{\mathcal{L}_2} \sim H^2\eta^2\frac{(\partial_\mu\pi)^2}{f^4} \sim \frac{H^4}{f^4}. \tag{9}$$

If we take $H > f$ the Goldstone boson ceases to be a weekly coupled degree of freedom and UV completion is required. In case of axion at high energies the Pecceiï£¡Quinn $U(1)$ symmetry is restored and the theory is described by the heavy complex scalar field with mass $\sim H$ (thermal).

3 Problems for Aspects of Eternal Inflation (Susskind)

Problem 2

We start with the de Sitter space in global slicing

$$ds^2 = -dt^2 + \frac{\cosh^2\chi}{H^2}(d\chi^2 + \sin^2\chi d\Omega^2). \tag{10}$$

It is obvious that in order to get the angular part of the interval to be given by $r^2 d\Omega$ one needs to make the following change of space coordinates

$$(Hr)^2 = \sin^2\chi \, \cosh^2(Ht). \tag{11}$$

Moreover, the absence of the off-diagonal metric components, namely g_{0r}, and the time independence of the remaining components requires (11) to be suplmented by the following transformation of the time-like coordinate

$$\tanh^2(H\tau) = \frac{\tanh^2 Ht}{\cos^2\chi}. \tag{12}$$

As a result of (11) and (12), the interval from (10) takes the following form

$$ds^2 = -(1 - H^2r^2)d\tau^2 + \frac{dr^2}{1 - H^2r^2} + r^2 d\Omega^2. \tag{13}$$

4 Mechanisms for Inflation (Silverstein)

Problem 3 (the solution follows quite closely hep-th/0404084)

The equations of motion for the homogeneous field in DBI inflation is given by

$$\ddot{\phi} - \frac{6}{\phi}\dot{\phi}^2 + \frac{4\phi^3}{\lambda} + \frac{3H}{\gamma^2}\dot{\phi} + \frac{V'}{\gamma^3} = 0, \tag{14}$$

where γ is given by

$$\gamma \equiv \frac{1}{\sqrt{1 + \frac{\lambda}{\phi^4}\dot{\phi}^2}}. \tag{15}$$

The interesting to us limit corresponds to the $\gamma \gg 1$, since in other case the model reduces to the theory of the scalar field with ordinary kinetic term and the effective potential $V + \phi^4/\lambda$. Notice that in this limit the last two terms of (14) are subdominant.

The Friedmann equation reads as follows

$$H^2 = \frac{8\pi G}{3}(\phi^4\gamma/\lambda - V). \tag{16}$$

By taking the time derivative of this equation and taking into account (14) we arrive at

$$\dot{\phi} = -\frac{1}{4\pi G}\frac{H'}{\gamma} \Rightarrow \gamma^2 = 1 - (4\pi G)^{-2}\frac{\lambda}{\phi^4}H'^2. \tag{17}$$

In order to have an inflation the potential energy should dominate over the kinetic energy (it easy to make sure that in this regime the the equation of state is almost the same as for the cosmological constant). Thus the Friedmann equation given above reduces to

$$H^2 = \frac{8\pi G}{3}(-V). \tag{18}$$

Differentiating this equation we find that the condition of the domination of potential energy gives (in Plank units)

$$\frac{\sqrt{\lambda}(-V)^{3/2}}{\phi^2 V'} \gg 1. \tag{19}$$

While the $\gamma \gg 1$ condition reduces to

$$\frac{\lambda V'^2}{-V\phi^4} \gg 1, \tag{20}$$

In contrary to the usual slow-roll conditions. It must be pointed out that the terms in the expansion of the action in perturbations are enhanced by factors of γ, which will lead to the strong non-Gaussianity.

Group 8.

Robustness of GR; Problem #3

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{\Lambda^3}(\partial_\mu \pi)^2 \Box \pi + \frac{1}{M_{pl}} \pi T \qquad (*)$$

From this Lagrangian we get the following equations of motion

$$\Box \pi + \frac{2}{\Lambda^3}\left((\Box \pi)^2 - (\partial_\mu \partial_\nu \pi)^2\right) + \frac{1}{M_{pl}} T = 0. \qquad (1)$$

$$\Downarrow$$

$$\Delta \pi + \frac{2}{\Lambda^3}\left((\Delta \pi)^2 - (\partial_i \partial_j \pi)^2\right) = \frac{M}{M_{pl}} \delta^{(3)}(\vec{x})$$

$$\partial_i \partial_j \pi = \left(\pi'' - \frac{\pi'}{r}\right)\frac{x_i x_j}{r^2} + \frac{\pi'}{r}\delta_{ij}$$

$$\Downarrow$$

$$\Delta \pi = \pi'' + \frac{2\pi'}{r} \;;$$

$$(1) \longrightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \partial_r \pi\right) + \frac{4}{\Lambda^3} \cdot \frac{1}{r^2}\frac{\partial}{\partial r}\left(r(\pi')^2\right) = \frac{M}{M_{pl}} \delta^{(3)}(\vec{x}).$$

After taking the volume integral from both sides:

$$r^2 \partial_r \pi + \frac{4}{\Lambda^3} r(\pi')^2 = \frac{M}{M_{pl}}, \qquad (2)$$

the Vainshtein radius $r_* \equiv \frac{1}{\Lambda}\left(\frac{M}{M_{pl}}\right)^{1/3}$

The solution of (2) that corresponds to the right asymptotics is:

$$\partial_r \pi = \frac{\Lambda^3}{8} \cdot \left(-r + r_* \sqrt{16\frac{r_*}{r} + \frac{r^2}{r_*^2}}\right) \Rightarrow \pi(r \gg r_*) = -\frac{M}{M_{pl}}\frac{1}{r}$$

and $\pi(r \ll r_*) = \frac{1}{4}\Lambda^{3/2}\left(\frac{M}{M_{pl}}\right)^{1/2} \cdot r^{1/2}$

Now, Let's find the ~~small~~ dynamics of the small fluctuations ~~around~~ this classical background t

$\pi = \phi^{cl} + \varphi$.

$$(*) \longrightarrow \mathcal{L} = -\frac{1}{2}(\partial_\mu \varphi)^2 \cdot \left[1 + \frac{4}{\Lambda^3} \Box \phi^{cl}\right] + \frac{2}{\Lambda^3} \partial_\mu \partial_\nu \phi^{cl} \partial_\mu \pi \partial_\nu \pi$$

- it is easy to see that the speed of sound for the perturbations (radial), is given by: at large distances $r \gg r_*$ we have:

$$C_s^2(r \gg r_*) = 1 + \frac{1}{\Lambda^3}\partial_r^2 \cdot (\phi^{cl}) = 1 + \frac{8}{\Lambda^3} \frac{M}{M_{Pl}} \frac{1}{r^3}$$

we see that modes propagate with the superluminal speed but the deviation from $C_s = 1$ is small at large distances.

However, the situation changes if we consider propagations well inside Vainshtein radius:

$$C_s^2(r \ll r_*) = \frac{4}{3} \Rightarrow C_s = \frac{2}{\sqrt{3}}$$

This are valid for wavelength $\lambda \ll r$.