Solutions to Homework from Maldacena

by Jolyon Bloomfield

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1 Problem #4

We want to evaluate the action

$$S_E = \frac{R_{AdS}^2}{16\pi G_N}\left[-\int_{\Sigma_4} d^4x\sqrt{g}(R+6) - 2\int_{\partial\Sigma_4} d^3x\sqrt{h}K\right] \tag{1}$$

for the Euclidean AdS metric

$$ds^2 = d\rho^2 + \sinh^2(\rho)d\Omega_3^2. \tag{2}$$

The Ricci scalar $R = -12$ for this metric.

Choose a volume to integrate over of a 4-ball of radius ρ_c. The first integral is

$$\begin{aligned} -\int_{\Sigma_4} d^4x\sqrt{g}(R+6) &= 6\int_0^{\rho_c} d\rho\sinh^3(\rho)\int d\Omega_3 && (3)\\ &= 6\left(-\frac{3}{4}\cosh(\rho_c) + \frac{1}{12}\cosh(3\rho_c) + \frac{2}{3}\right)2\pi^2 && (4)\\ &= \pi^2\left(-9\cosh(\rho_c) + \cosh(3\rho_c) + 8\right) && (5)\end{aligned}$$

The integral over the angles is the surface area of a sphere of radius 1, which is $2\pi^2$.

The second integral needs the extrinsic curvature tensor. To evaluate that, we need the normal:

$$\vec{n} = -\frac{\partial}{\partial\rho} \tag{6}$$

Note that this is normalized ($n_a n^a = 1$), and points inwards from the boundary. We also need the induced metric on the boundary, which is given by $\rho = \rho_c$, so

$$d\sigma^2 = \sinh^2(\rho_c)d\Omega_3^2. \tag{7}$$

This gives the induced metric h_{ab}. The extrinsic curvature tensor is given by

$$K_{ab} = P_a^c P_b^d \nabla_{(c}n_{d)} \tag{8}$$

where $P_{ab} = g_{ab} - n_a n_b$ is the projection tensor. The nonvanishing terms are

$$K_{aa} = \nabla_a n_a = -\Gamma^r{}_{aa}n_\rho = \Gamma^\rho{}_{aa} \tag{9}$$

where $a = \theta, \phi, \psi$, the three angular variables. Of course, the derivatives vanish, leaving only the connection coefficient, and n_ρ is the only nonzero component of the normal. Now,

$$\Gamma^\rho{}_{\theta\theta} = -\cosh(\rho)\sinh(\rho) = \frac{\Gamma^\rho{}_{\phi\phi}}{\sin^2(\theta)} = \frac{\Gamma^\rho{}_{\psi\psi}}{\sin^2(\theta)\sin^2(\phi)}. \tag{10}$$

Then, $K = h^{ab}K_{ab}$ evaluates to

$$K = -3\coth(\rho_c). \tag{11}$$

The integral over K is then

$$-2\int_{\partial\Sigma_4} d^3x\sqrt{h}K = 6\coth(\rho_c)\sinh^3(\rho_c)\int d\Omega_3 \tag{12}$$
$$= 12\pi^2\cosh(\rho_c)\left(\cosh^2(\rho_c) - 1\right). \tag{13}$$

We can then calculate the action

$$S_E = \frac{\pi R_{AdS}^2}{16G_N}\left[-9\cosh(\rho_c) + \cosh(3\rho_c) + 8 + 12\cosh(\rho_c)\left(\cosh^2(\rho_c) - 1\right)\right] \tag{14}$$
$$= \frac{\pi R_{AdS}^2}{16G_N}\left[-21\cosh(\rho_c) + \cosh(3\rho_c) + 8 + 12\cosh^3(\rho_c)\right] \tag{15}$$

Next, use the identity

$$\cosh^3(x) = \frac{1}{4}\left(\cosh(3x) + 3\cosh(x)\right). \tag{16}$$

$$S_E = \frac{\pi R_{AdS}^2}{16G_N}\left[-9\cosh(\rho_c) + \cosh(3\rho_c) + 8 + 12\cosh(\rho_c)\left(\cosh^2(\rho_c) - 1\right)\right] \tag{17}$$
$$= \frac{\pi R_{AdS}^2}{16G_N}\left[-12\cosh(\rho_c) + 4\cosh(3\rho_c) + 8\right] \tag{18}$$

Discarding the divergent piece of this, we have $\cosh(x) \to \exp(-x)/2$. The finite part of the Euclidean action is then

$$S_E = \frac{\pi R_{AdS}^2}{16G_N}\left[-6e^{-\rho_c} + 2e^{-3\rho_c} + 8\right]. \tag{19}$$

Then, for $\Psi = Z \sim \exp(-S_E)$, we have

$$Z \sim \exp\left\{\frac{\pi R_{AdS}^2}{8G_N}\left[3e^{-\rho_c} - e^{-3\rho_c} - 4\right]\right\}. \tag{20}$$

2 Problem #5

We want to start with the dS metric

$$ds^2 = -d\tau^2 + \cosh^2(\tau)d\Omega_3^2. \tag{21}$$

However, before putting it into the Euclidean action, we need to analytically continue to Euclidean time. Thus, let $\tau \to it$. The metric then becomes

$$ds^2 = dt^2 + \cos^2(t)d\Omega_3^2. \tag{22}$$

Now, we can evaluate the action

$$S_E = \frac{R_{AdS}^2}{16\pi G_N}\left[-\int_{\Sigma_4} d^4x\sqrt{g}(R+6) - 2\int_{\partial\Sigma_4} d^3x\sqrt{h}K\right] \tag{23}$$

using this metric. The Ricci scalar $R = 12$ for this metric (surprise surprise?).

Choose a volume to integrate over of a 4-ball of radius t_c. The first integral is

$$-\int_{\Sigma_4} d^4x\sqrt{g}(R+6) = -18\int_0^{t_c} dt\cos^3(t)\int d\Omega_3 \tag{24}$$

$$= -18\left(\frac{3}{4}\sin(t)+\frac{1}{12}\sin(3t)\right)2\pi^2 \tag{25}$$

$$= -36\pi^2\left(\frac{3}{4}\sin(t)+\frac{1}{12}\sin(3t)\right) \tag{26}$$

The integral over the angles is the surface area of a sphere of radius 1, which is again, $2\pi^2$.

The second integral needs the extrinsic curvature tensor. To evaluate that, we need the normal:

$$\vec{n} = -\frac{\partial}{\partial t} \tag{27}$$

Note that this is normalized ($n_a n^a = 1$), and points inwards from the boundary. We also need the induced metric on the boundary, which is given by $t = t_c$, so

$$d\sigma^2 = \cos^2(t_c)d\Omega_3^2. \tag{28}$$

This gives the induced metric h_{ab}. The extrinsic curvature tensor is given by

$$K_{ab} = P_a^c P_b^d \nabla_{(c}n_{d)} \tag{29}$$

where $P_{ab} = g_{ab} - n_a n_b$ is the projection tensor. The nonvanishing terms are

$$K_{aa} = \nabla_a n_a = -\Gamma^t{}_{aa}n_t = \Gamma^t{}_{aa} \tag{30}$$

where $a = \theta, \phi, \psi$, the three angular variables. Of course, the derivatives vanish, leaving only the connection coefficient, and n_t is the only nonzero component of the normal. Now,

$$\Gamma^t{}_{\theta\theta} = \cos(t)\sin(t) = \frac{\Gamma^T{}_{\phi\phi}}{\sin^2(\theta)} = \frac{\Gamma^t{}_{\psi\psi}}{\sin^2(\theta)\sin^2(\phi)}. \tag{31}$$

Then, $K = h^{ab}K_{ab}$ evaluates to

$$K = 3\tan(t). \tag{32}$$

The integral over K is then

$$-2\int_{\partial\Sigma_4} d^3x\sqrt{h}K = -6\tan(t_c)\cos^3(t_c)\int d\Omega_3 \tag{33}$$

$$= -12\pi^2\sin(t_c)\cos^2(t_c). \tag{34}$$

We can then calculate the action

$$S_E = \frac{R_{AdS}^2}{16\pi G_N}\left[-36\pi^2\left(\frac{3}{4}\sin(t_c)+\frac{1}{12}\sin(3t_c)\right)-12\pi^2\sin(t_c)\cos^2(t_c)\right] \tag{35}$$

$$= \frac{\pi R_{AdS}^2}{16G_N}\left[-27\sin(t_c)-3\sin(3t_c)-12\sin(t_c)(1-\sin^2(t_c))\right] \tag{36}$$

$$= \frac{\pi R_{AdS}^2}{16G_N}\left[-39\sin(t_c)-3\sin(3t_c)+12\sin^3(t_c)\right] \tag{37}$$

Now, we rotate back to the Minkowski action, by $t \to -i\tau$.

$$S = \frac{\pi R_{AdS}^2}{16G_N}\left[-39\sin(-i\tau_c)-3\sin(-3i\tau_c)+12\sin^3(-i\tau_c)\right] \tag{38}$$

$$= i\frac{\pi R_{AdS}^2}{16G_N}\left[39\sinh(\tau_c)+3\sinh(3\tau_c)+12\sinh^3(\tau_c)\right] \tag{39}$$

We want to then calculate $\Psi \sim e^{iS}$. Doing this, the factors of i multiply to give -1, but then what remains of iS is either divergent or decaying as $t_c \to \infty$. This can be understood as when we go out to infinite time, the volume of de Sitter space increases without bound and diverges, and so the action that we compute must diverge.

Solutions to Homework from Creminelli

by Jolyon Bloomfield

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1 Problem #1

2 Problem #2

In a generic model of inflation, assume a de Sitter space metric.

$$ds^2 = \frac{1}{\eta^2 H^2} \left(-d\eta^2 + dx^2\right) \tag{1}$$

This metric has the usual rotational and translational invariance, but it also has a dilation symmetry: $\tau = \eta/\lambda, y = x/\lambda$ leaves the metric invariant also.

Now, if we have a function $\zeta(x)$, then the Fourier transformed version is

$$\zeta_k = \int d^3x e^{ik\cdot x} \zeta(x). \tag{2}$$

Under a dilation, $\zeta(x) = \zeta(y\lambda)$, and so

$$\zeta_k = \int d^3y e^{ik\cdot y} \zeta(y\lambda) = \frac{1}{\lambda^3} \int d^3x e^{ik\cdot x/\lambda} \zeta(x) \tag{3}$$

$$\zeta_{k'\lambda} = \frac{1}{\lambda^3} \int d^3x e^{ik'\cdot x} \zeta(x) = \frac{1}{\lambda^3} \zeta_{k'} \tag{4}$$

Inverting this, we have $\zeta_{k'/\lambda} = \lambda^3 \zeta_{k'}$ under the dilation.

Then, in an n-point correlation function, we have

$$\left\langle \zeta_{k_1'/\lambda} \cdots \zeta_{k_n'/\lambda} \right\rangle = \left\langle \zeta_{k_1'} \cdots \zeta_{k_n'} \right\rangle \lambda^{3n} \tag{5}$$

under a dilation.

Now, translational invariance implies conservation of momentum, which gives us the usual delta function of momentum conservation, and so we have

$$\langle \zeta_{k_1} \cdots \zeta_{k_n} \rangle = (2\pi)^3 \delta^3 \left(\sum_i^n \vec{k}_i \right) F(k_i) \tag{6}$$

Under a dilation, the delta function scales as $\delta^3(k/\lambda) = \lambda^3 \delta^3(k)$. Thus, $F(k_i/\lambda)$ must scale as $\lambda^{3(n-1)}$, and so F must be a homogeneous function of k_i of degree $-3(n-1)$ in order to get the correct scaling. Note that a homogeneous function of degree n is one for which $f(\lambda x) = \lambda^n f(x)$.

F may also have a dependence on $|\vec{k}_i|\eta$, but we see that this dependence has to vanish, because ζ_k must be time independent on scales outside the horizon.

3 Problem #4

We can see that photons are not produced during inflation because the action of the electromagnetic field is conformally invariant.

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{4}g^{\mu\lambda}g^{\nu\sigma}F_{\mu\nu}F_{\lambda\sigma}\right\} \tag{7}$$

Under a conformal transformation, $g_{\mu\nu} \to \Omega^2(x^a)g_{\mu\nu}$, we have $\sqrt{-g} \to \Omega^4(x^a)\sqrt{-g}$ and $g^{\mu\lambda} \to \Omega^{-2}(x^a)g^{\mu\lambda}$. The Faraday tensor is unchanged, and so it can be seen that the action is invariant under this transformation.

Now, the metric describing the inflationary phase is given by an FRW metric

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2. \tag{8}$$

Changing to conformal time, $d\eta = dt/a$, this becomes conformally flat

$$ds^2 = -a^2(\eta)\left[d\eta^2 + d\vec{x}^2\right]. \tag{9}$$

Thus, we see that the equations of motion for the electromagnetic field are the same in an unperturbed FRW metric as they are in Minkowski space. As such, the metric does not couple to the electromagnetic field, and so there is no amplification of the electromagnetic fields under inflation.

We can make this qualitative argument more rigorous by noting that the EM vacuum must also be conformally invariant. Thus, the evolution of the vacuum in Minkowski space is the same as the evolution of the vacuum in the FRW space. Thus, the number operator acting on the vacuum after propagating it forward in time will still annihilate the vacuum.

Note that for a perturbed FRW metric, in particular, with tensor perturbations, the metric is no longer conformally flat. This means that the tensor perturbations will couple to the electromagnetic field, which can cause photon production. However, the rate for this will be very heavily suppressed.

Solutions to Homework from Susskind

by Jolyon Bloomfield

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0 Problem #0

I did this exercise for my own benefit.

Problem: derive the metric in global slicing on de Sitter space.

$$ds^2 = R^2 \left[-d\tau^2 + \cosh^2(\tau) d\Omega_3^2\right] \tag{1}$$

Solution: start with the hyperboloid in 5d:

$$-T^2 + X_0^2 + X_1^2 + X_2^2 + X_3^2 = R^2 \tag{2}$$

We want to parameterize the T coordinate based on some hyperbolic angle, so let

$$T = R \sinh(\tau). \tag{3}$$

Now, we want the rest of the coordinates to sum as follows.

$$X_0^2 + X_1^2 + X_2^2 + X_3^2 = R^2 \cosh^2(\tau) \tag{4}$$

We also want to do this with only 3 coordinates. So, let's let the X_i coordinates represent four-dimensional spherical polar coordinates in (r, θ, ϕ, ψ), and set $r = R\cosh(\tau)$. Then the Minkowski metric in 5d is

$$\begin{aligned} ds^2 &= -dT^2 + dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 && (5) \\ &= -dT^2 + dr^2 + r^2 d\Omega_3^2. && (6) \end{aligned}$$

Then, substituting in for dT and dr, we have

$$ds^2 = R^2 \left[-d\tau^2 + \cosh^2(\tau) d\Omega_3^2\right] \tag{7}$$

as desired.

1 Problem #1

Problem: derive the metric in flat slicing on de Sitter space.

$$ds^2 = R^2 \left[-d\tau^2 + e^{2\tau} dx^i dx^i\right] \tag{8}$$

Solution: here we're going to want three coordinates to form the flat three-metric, so let X_1, X_2, and X_3 be 3d spherical polar coordinates in (r', θ, ϕ). Then the hyperboloid in 5d can be written as

$$-T^2 + X_0^2 + r'^2 = R^2. \tag{9}$$

Motivated by the form of the solution, we're going to want to choose $r' = Re^{\tau}r$ eventually. Furthermore, taking the suggestion that we want to make a null slicing of the hyperboloid, we need $T + X_0 = \text{const}$, where this constant will have to depend on our final coordinate τ and the length scale R. We then need to split this into two equations, parameterizing the solution in terms of τ and r alone. When $r = 0$, we know that the parametrization takes the form

$$T = R\sinh(\tau) \tag{10}$$

$$X_0 = R\cosh(\tau), \tag{11}$$

and so this will be our starting point. Note that this means that τ is the proper time along the edge of the hyperboloid, as drawn in the diagram in class. We can add the same function to T and subtract it from X_0, which preserves the form of $T + X_0$. This function can be a function of τ, r, and R. So, let

$$T = R\sinh(\tau) + Rf(r, \tau, R) \tag{12}$$

$$X_0 = R\cosh(\tau) - Rf(r, \tau, R). \tag{13}$$

Next, we look at the hyperboloid itself to find the form of f that we require.

$$R^2 = -T^2 + X_0^2 + r'^2 \tag{14}$$

$$R^2 = -\left(R\sinh(\tau) + Rf\right)^2 + \left(R\cosh(\tau) - Rf\right)^2 + R^2r^2e^{2\tau} \tag{15}$$

$$1 = 1 - 2f\left(\sinh(\tau) + \cosh(\tau)\right) + r^2e^{2\tau} \tag{16}$$

$$2fe^{\tau} = r^2e^{2\tau} \tag{17}$$

Thus, we find we need

$$f = \frac{r^2}{2}e^{\tau}. \tag{18}$$

Next, we look at the 5d Minkowski metric, and re-express it in terms of our new coordinates.

$$ds^2 = -dT^2 + dX_0^2 + dX_1^2 + dX_2^2 + dX_3^2 \tag{19}$$

$$= -dT^2 + dX_0^2 + dr'^2 + r'^2d\Omega_2^2 \tag{20}$$

$$= R^2\left[-d\tau^2 + e^{2\tau}\left(dr^2 + r^2d\Omega_2^2\right)\right] \tag{21}$$

Ok, so I've skipped a few steps in here, but they're uninteresting, and you can check that they yield the result. There's a lot of cancelations to deal with.

2 Problem #2

Problem: derive the metric in static slicing on de Sitter space.

$$ds^2 = R^2\left[-(1-r^2)d\tau^2 + \frac{dr^2}{1-r^2} + r^2d\Omega_2^2\right] \tag{22}$$

Solution: once again, we want something that resembles 3d spherical coordinates, so let X_1, X_2, and X_3 be 3d spherical polar coordinates in (r, θ, ϕ). Then the hyperboloid in 5d can be written as

$$-T^2 + X_0^2 + r^2 = R^2. \tag{23}$$

This time though, we want $T/X_0 = \tanh(\tau)$ based on the geometry of the slicing, where τ is the hyperbolic angle of rotation. A simple set of coordinates that satisfies the hyperboloid equation is then

$$T = \sqrt{R^2 - r^2}\sinh(\tau) \tag{24}$$

$$R = \sqrt{R^2 - r^2}\cosh(\tau). \tag{25}$$

The r coordinate is the portion of the R^2 that isn't taken up by X_0 and T. On the hyperboloid picture, $r = 0$ when you're on the hyperboloid itself, whereas when $r = R$, T and X_0 are vanishing - this is on the equator.

When you work through the coordinate transformation from the metric

$$ds^2 = -dT^2 + dX_0^2 + dr^2 + r^2 d\Omega_2^2, \tag{26}$$

you end up with the following metric.

$$ds^2 = -(R^2 - r^2)d\tau^2 + \left(\frac{r^2}{R^2 - r^2} + 1\right) dr^2 + r^2 d\Omega_2^2 \tag{27}$$

A final change of coordinates, $r = Rr'$, yields our final result

$$ds^2 = R^2 \left[-(1 - r'^2)d\tau^2 + \frac{1}{1 - r'^2} dr'^2 + r'^2 d\Omega_2^2\right]. \tag{28}$$

3 Problem #3

Problem: show that the spatial metric here is that for a hemisphere on a 3-sphere.

$$ds^2 = R^2 \left[\frac{1}{1 - r^2} dr^2 + r^2 d\Omega_2^2\right] \tag{29}$$

Solution: the usual metric on a 3-sphere is given by

$$\begin{aligned} ds^2 &= R^2 d\Omega_3^2 && (30) \\ &= R^2 d\theta^2 + R^2 \sin^2(\theta) d\phi^2 + R^2 \sin^2(\theta) \sin^2(\phi) d\psi^2 && (31) \\ &= R^2 d\theta^2 + R^2 \sin^2(\theta) d\Omega_2^2. && (32) \end{aligned}$$

We see by observation that we want $r^2 = R^2 \sin^2(\theta)$. Taking $r = R\sin(\theta)$, we can compute

$$\begin{aligned} dr &= -R\cos(\theta) d\theta && (33) \\ dr^2 &= R^2(1 - \sin^2(\theta)) d\theta^2 && (34) \\ d\theta^2 &= \frac{dr^2}{1 - r^2/R^2} && (35) \end{aligned}$$

Again, taking a final transformation $r \to Rr'$, we have the explicit coordinate transformation that takes

$$ds^2 = R^2 \left[\frac{1}{1 - r'^2} dr'^2 + r'^2 d\Omega_2^2\right] \tag{36}$$

into

$$ds^2 = R^2 d\Omega_3^2, \tag{37}$$

and so this is the metric on a hemisphere. The way to tell it's a hemisphere instead of the entire sphere is that there is a coordinate singularity at $r = R$ ($r' = 1$), which corresponds to $\theta = \pi/2$, and so θ can only range from 0 to $\pi/2$, or from $\pi/2$ to π.

4 Problem #4

Problem: write down the rate equations for transitions between the vacua. Show that the transition matrix can be made symmetric, that it has a zero eigenvalue, that all the other eigenvalues are negative, and find the zero eigenvector.

Solution: the transition equations are

$$\Delta P_a = -\sum_b \gamma_{ba} P_a + \sum_b \gamma_{ab} P_b \tag{38}$$

where the sums are over $b \neq a$, and Einstein summation convention is not implied.

We can write the transition probabilities as

$$\gamma_{ab} = M_{ab} e^{S_a} \tag{39}$$

where M_{ab} is a symmetric matrix element, and the exponential represents the phase space. If we also redefine the probabilities based on their phase space by $P_a = \exp(S_a/2)\phi_a$, then the transition equations can be written

$$e^{S_a/2}\Delta\phi_a = -\sum_b M_{ba} e^{S_b} e^{S_a/2}\phi_a + \sum_b M_{ab} e^{S_a} e^{S_b/2}\phi_b \tag{40}$$

$$\Delta\phi_a = -\sum_b M_{ba} e^{S_b}\phi_a + \sum_b M_{ab} e^{(S_a+S_b)/2}\phi_b. \tag{41}$$

We can write this as a matrix equation as

$$\Delta\vec{\phi} = A\vec{\phi} \tag{42}$$

where the matrix A is given by

$$A = \begin{pmatrix} -\sum_b M_{b1} e^{S_b} & M_{12} e^{(S_1+S_2)/2} & \cdots \\ M_{21} e^{(S_2+S_1)/2} & -\sum_b M_{b2} e^{S_b} & \\ \vdots & & \ddots \end{pmatrix} \tag{43}$$

which we see is explicitly symmetric, given that M_{ab} is symmetric.

We can see that there is a zero eigenvalue by seeing that the matrix is degenerate. The quickest way to see this is that the rows are linearly dependent. In particular, multiply the second row by $M_{12}\exp[(S_2 - S_1)/2]$, add it to the first row, and repeat for all other rows (with appropriate factors), and the first row will vanish. Thus, the rows are linearly dependent, which implies the matrix is degenerate, which in turn implies that there is a zero eigenvalue.

The next step is to find the eigenvector for the zero eigenvalue. Rather than searching for the eigenvector the usual way, I'll just show that the eigenvector that Susskind gave to us does indeed solve the system. The suggestion was to try $P_a = \exp(S_a)$, or $\phi_a = \exp(S_a/2)$. We look at a single entry, $\Delta\phi_a$.

$$\Delta\phi_a = -\sum_b M_{ba} e^{S_b}\phi_a + \sum_b M_{ab} e^{(S_a+S_b)/2}\phi_b \tag{44}$$

$$= -\sum_b M_{ba} e^{S_b+S_a/2} + \sum_b M_{ab} e^{S_a/2+S_b} \tag{45}$$

$$= 0 \tag{46}$$

Still to be done: show that the other eigenvalues are all negative.

5 Problem #4

Problem: Consider the case of three vacua labeled 0; 1 and 2. Assume 0 is terminal. Write down the transition matrix. Show that it has a zero eigenvalue and two negative ones. Find the eigenvector with zero eigenvalue. Show that the probabilities of finding the 1 and 2 vacua decrease with time but that the number of these vacua increases with time. Assume the elements of the transition matrix are small.

Solution: the transition equations are

$$\Delta P_a = -\sum_b \gamma_{ba} P_a + \sum_b \gamma_{ab} P_b \tag{47}$$

where the sums are over $b \neq a$, and Einstein summation convention is not implied. For vacua 0, $\gamma_{b0} = 0$.

We can write this as a matrix equation as

$$\Delta \vec{P} = A \vec{P} \tag{48}$$

where the matrix A is given by

$$A = \begin{pmatrix} 0 & \gamma_{01} & \gamma_{02} \\ 0 & -\gamma_{01} - \gamma_{21} & \gamma_{12} \\ 0 & \gamma_{21} & -\gamma_{02} - \gamma_{12} \end{pmatrix}. \tag{49}$$

This obviously has a zero eigenvalue. For the other two eigenvalues, look at the characteristic polynomial.

$$(-\lambda)\left[(-\gamma_{01} - \gamma_{21} - \lambda)(-\gamma_{02} - \gamma_{12} - \lambda) - \gamma_{12}\gamma_{21}\right] = 0 \tag{50}$$

Discarding the zero eigenvalue, we have

$$\lambda^2 + (\gamma_{01} + \gamma_{21} + \gamma_{02} + \gamma_{12})\lambda + (\gamma_{01} + \gamma_{21})(\gamma_{02} + \gamma_{12}) - \gamma_{12}\gamma_{21} = 0. \tag{51}$$

We know that a system with $(x-a)(x-b) = 0$ gives $x^2 - (a+b)x + ab = 0$. Given that all γ_{ab} are positive numbers, we have that $a+b$ is negative, so the sum of the two remaining eigenvalues $a+b$ must be negative. Similarly, we see that ab is positive. Thus, assuming the eigenvalues are real, they must both be negative. If the eigenvalues are complex, then they must be complex conjugates of each other. In that case, their sum will be twice the real component of the eigenvalue, which then must be negative. In either case, the real component of the eigenvalues is negative.

The zero eigenvector can be guessed by inspection.

$$\vec{P} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{52}$$

This corresponds to the entire multiverse being in the "dead" regime.

The other two eigenvectors will correspond to a component having a probability of living in vacuums 1 and 2. With each successive time step, each of these will decrease, as the eigenvalues for each of these two vectors is negative. Consider a state consisting of one of these eigenvalues. The evolution of the probability of having this state is given by

$$P(n+1) = (1-\lambda)P(n) \tag{53}$$

where λ is the eigenvalue. In terms of the original state, we then have

$$P(n) = (1-\lambda)^n P(0) \tag{54}$$

and so the probability of being in this eigenstate is decreases with each time step. However, because each cell is splitting into 8 each time, the total number of cells in this state is given by

$$N = (1-\lambda)^n 8^n. \tag{55}$$

Again, so long as the eigenvalue $\lambda < 7/8$, N will grow, even though the probability decreases. The eigenvalues λ scale linearly with the transition probabilities γ_{ab}, so for sufficiently small γ_{ab}, the absolute value of the eigenvalues will be less than 7/8.

The analysis of the other eigenvector is the same, albeit with a slightly different rate. Thus, we see that the probability of being in either vacuum decreases, while the total number of cells in that vacuum increases, for sufficiently small γ_{ab}.

Solutions to Homework from Silverstein

by Jolyon Bloomfield

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1 Problem #1

1.1 Part a)

The number of e-folds is given by

$$N_e = \ln\left(\frac{a(t_{end})}{a(t_{start})}\right) \tag{1}$$

$$= \int_{t_{start}}^{t_{end}} H dt \tag{2}$$

$$= \int_{t_{start}}^{t_{end}} \frac{H}{\dot{\phi}} d\phi \tag{3}$$

Taking the equation of motion for ϕ,

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4}$$

and approximating that $\ddot{\phi} \ll H\dot{\phi}$ (equivalent to one of the slow roll conditions), we have

$$\dot{\phi} \approx -\frac{V'}{3H}. \tag{5}$$

Next, we want $\epsilon \ll 1$ (the other slow roll condition), and so $H^2 \approx V/3m_p^2$. Thus, the number of e-folds is approximately

$$N_e = \int_{t_{end}}^{t_{start}} \frac{1}{m_p^2} \frac{V}{V'} d\phi. \tag{6}$$

Note that

$$\frac{\eta^{-1/2}}{m_p} \approx \frac{1}{m_p^2}\frac{V}{V'} = \frac{\phi}{pm_P^2} \tag{7}$$

with $V = \lambda\mu^{4-p}\phi^p$. Then

$$N_e = \frac{\phi^2(t_{start})}{2pm_P^2} \tag{8}$$

under the assumption that the field has moved a long way over the course of inflation ($\Delta\phi \sim \phi(t_{start})$). The field range is then

$$\Delta\phi \approx m_P\sqrt{2pN_e}. \tag{9}$$

Still to do: determine the condition on λ and μ by requiring the power spectrum to have the COBE normalization.

2 Problem #2

2.1 Part a)

We begin by assuming we are working with a simple product space, with

$$ds^2 = g^{(4)}_{ab}(x^c)dx^a dx^b + h_{\mu\nu}(y^\alpha)dy^\mu dy^\nu \tag{10}$$

as our metric. For this simple space, the terms in the Ricci scalar associated with the spacetime and the internal dimensions completely decouple, and so

$$R = R^{(4)} + R^{int}. \tag{11}$$

The internal space will then create an effective potential term in the action. For simple internal spaces, we'll assume that we've compactified on a torus, and use $R^{int} = 0$.

For our action, we take just the Einstein-Hilbert term in the metric.

$$S = \int d^D x\sqrt{-g}\frac{e^{-2\Phi}}{\alpha'^{(D-2)/2}}R \tag{12}$$

Assuming that the dilaton has no dependence on the internal dimensions, we can split the integral into a spacetime integral, and an internal space integral.

$$S = \int d^4 x\sqrt{-g^{(4)}}\frac{e^{-2\Phi}}{\alpha'^{(D-2)/2}}\left(R^{(4)}\right)\int d^{D-4}y\sqrt{h} \tag{13}$$

Recall that $R^{(4)}$ doesn't depend on the internal degrees of freedom either (no warping here, folks). Now, the integral

$$\int d^{D-4}y\sqrt{h} = V_X(x^a) \tag{14}$$

where V_X is the volume of the internal space. Note that it can have a spacetime dependence, as the limits on the integral can be dependent upon the spacetime position (for a simple example, think of a torus whose radii vary with time). Our action in string frame is then

$$S = \int d^4 x\sqrt{-g^{(4)}}\frac{V_0 V_X e^{-2\Phi}}{V_0\alpha'^{(D-2)/2}}R^{(4)}. \tag{15}$$

The importance of scaling out V_0 here is so that we can make a conformal transformation using a dimensionless parameter. At this stage, we identify

$$\frac{V_0}{\alpha'^{(D-2)/2}} = \frac{m_p^2}{2} \tag{16}$$

for simplicity (although that this is the case will not be evident until we transform to Einstein frame).

$$S = \int d^4 x\sqrt{-g^{(4)}}\frac{V_X}{V_0}e^{-2\Phi}\frac{m_p^2}{2}R^{(4)}. \tag{17}$$

Next, we want to transform to Einstein frame. The Einstein frame metric is given by

$$\gamma_{ab} = g^{(4)}_{ab}\frac{V_X}{V_0}e^{-2\Phi} = g^{(4)}_{ab}\Theta \tag{18}$$

where $\Theta = V_X/V_0\exp(-2\Phi)$ for simplicity for the moment. Under this transformation, $\sqrt{-g^{(4)}} = \sqrt{-\gamma}\Theta^{-2}$. The Ricci scalar transforms as follows. You can derive this from scratch, or look it up in the appendices of Wald's GR book, for example (or any other good GR book, and I suspect most string theory books too).

$$R^{(4)}[g^{(4)}_{ab}] = \Theta\left\{R^{(4)}[\gamma] - \frac{3}{2\Theta^2}(\nabla^a\Theta)(\nabla_a\Theta) - 6\nabla^a\nabla_a\ln\Theta^{-1/2}\right\} \tag{19}$$

Covariant derivatives here are those associated with the metric γ_{ab}, and are contracted with this metric also. Putting this into the action, we have

$$S = \int d^4x\sqrt{-\gamma}\Theta^{-2}\Theta\frac{m_p^2}{2}\Theta\left(R^{(4)}[\gamma] - \frac{3}{2\Theta^2}(\nabla^a\Theta)(\nabla_a\Theta) - 6\nabla^a\nabla_a\ln\Theta^{-1/2}\right) \tag{20}$$

$$= \int d^4x\sqrt{-\gamma}\frac{m_p^2}{2}\left(R^{(4)}[\gamma] - \frac{3}{2\Theta^2}(\nabla^a\Theta)(\nabla_a\Theta) - 6\nabla^a\nabla_a\ln\Theta^{-1/2}\right). \tag{21}$$

Here, we can see that the $\nabla^2\ln\Theta^{-1/2}$ term is a total derivative, and so we drop the term. At this point, we are in the Einstein frame (the coefficient of the Ricci scalar is $m_p^2/2$). All that now remains is to canonically normalize the kinetic term. Let's let $\exp(2\sigma) = V_X/V_0$ for the moment, so that $\Theta = \exp(2(\sigma - \Phi))$.

$$S = \int d^4x\sqrt{-\gamma}\left(\frac{m_p^2}{2}R^{(4)}[\gamma] - \frac{1}{2}6m_p^2\nabla^a(\sigma-\Phi)\nabla_a(\sigma-\Phi)\right) \tag{22}$$

To canonically normalize σ and Φ, define $\sigma_X = \sqrt{6}m_p\sigma$, and $\Phi_X = \sqrt{6}m_p\Phi$.

$$S = \int d^4x\sqrt{-\gamma}\left(\frac{m_p^2}{2}R^{(4)}[\gamma] - \frac{1}{2}\nabla^a(\sigma_X-\Phi_X)\nabla_a(\sigma_X-\Phi_X)\right) \tag{23}$$

At this stage, if we drop the dilaton (let $\Phi_X \to 0$), σ_X is canonically normalized, and we have

$$V_X = V_0 e^{\sqrt{2/3}\sigma_X/m_p} \tag{24}$$

so that the factor c_X mentioned in the problem takes the value of $\sqrt{2/3}$.

Keeping the dilaton around, however, we can perform a field redefinition to diagonalize the kinetic term. Let $\psi = \sigma_X - \Phi_X$, and $\varphi = \sigma_X + \Phi_X$. Then φ is an auxiliary field, and the action becomes

$$S = \int d^4x\sqrt{-\gamma}\left(\frac{m_p^2}{2}R^{(4)}[\gamma] - \frac{1}{2}\nabla^a\psi\nabla_a\psi\right). \tag{25}$$

Although φ doesn't appear in this action, it may make an appearance in the other terms in the action that we haven't looked at. For our purposes though, we only ever had $V_X\exp(-2\Phi)$, and so we were only going to end up with one field making an appearance.

2.2 Part b)

A negative mass particle has a variety of problems associated with it. In particular, they can pair produce, which leads to an instability by which the particles proliferate. If you throw one of these particles into a black hole, the mass of the black hole decreases, and so the area of the event horizon decreases, and given that the entropy of the black hole is proportional to the area of the event horizon, the entropy decreases. Thus, negative mass particles violate the second law of thermodynamics.

On the other hand, an orientifold is better behaved. Even though orientifolds source negative gravitational potential, they cannot proliferate. The presence of an orientifold maps $x_\perp \leftrightarrow -x_\perp$, and so it changes the asymptotics of the space completely. It essentially turns the spacetime into a cone, with the orientifold sitting at the tip of the cone. The number of orientifolds in a spacetime is restricted because of the way the asymptotics of the spacetime are effected in this manner, so these objects can't proliferate, and thus don't have the instability associated with negative mass particles.

I'm not so sure on what happens when you throw an orientifold (an extended space-filling object) into a black hole.

2.3 Part c)

We have terms in the action of the form

$$-|dB|^2 - |dC_p + B \wedge dC_{p-2}|^2. \tag{26}$$

Here, we assume that there is only one of the second term, for a specific p. We would like to gauge the two-form B with a symmetry $B \to B + d\Lambda_1$. The first term is obviously invariant under this transformation, as $d^2\Lambda_1 = 0$.

To make the second term invariant under this symmetry, consider $C_p \to C_p + f_p$ under the action of the symmetry, while leaving C_{p-2} a a singlet under the transformation. Then

$$dC_p + B \wedge dC_{p-2} \to dC_p + df_p + B \wedge dC_{p-2} + d\Lambda_1 \wedge dC_{p-2}. \tag{27}$$

We see that we need

$$df_p = -d\Lambda_1 \wedge dC_{p-2}. \tag{28}$$

A choice of f_p that satisfies this criteria is

$$f_p = -\Lambda_1 \wedge dC_{p-2} + dg_{p-1} \tag{29}$$

where g_{p-1} is an arbitrary $p-1$-form.

2.4 Part d)

The setup for this problem is a little confusing. In particular, the metric given is entirely a red herring - you cannot use it to calculate the four-dimensional Planck mass, as far as I know. I'll begin by describing how we're going to set up and attack the problem before doing the calculation.

The idea is that you have a warped throat with a stack of D3-branes down the IR end of the throat. At the UV end of the throat, there is some compactification that isn't too interesting; we're not going to go near it in this problem. It's probably a Calabi-Yau of some sort. We're looking at the dynamics of a D3-brane moving in this warped throat. The position of the brane is given by a coordinate r, and a bunch of angular coordinates that we won't concern ourselves with. In the throat, the metric is

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + ds^2_{X_5} \tag{30}$$

where $ds^2_{X_5}$ is the metric on the angular coordinates, which is assumed to not depend on t or $\vec{x}$. It may depend on r, but we'll ignore this, because all we are trying to do is to calculate the effective coefficient of $R^{(4)}$ in the action, and any r dependence in $ds^2_{X_5}$ won't change this.

What we want to do is to start with the action

$$S = \int d^{10}x\sqrt{-g}\frac{e^{-2\Phi}}{\alpha'^4}R^{(10)} \tag{31}$$

and perform a dimensional reduction by integrating over the six internal dimensions. To perform the dimensional reduction, we assume the metric

$$ds^2 = \frac{r^2}{R^2}g^{(4)}_{\mu\nu}(x^\alpha)dx^\mu dx^\nu + \frac{R^2}{r^2}dr^2 + ds^2_{X_5} \tag{32}$$

and calculate the Ricci scalar, $R^{(10)}$. The only term we are concerned with is the term containing $R^{(4)}[g_{\mu\nu}]$. Any other terms will form effective potential terms in the action after integrating over the internal dimensions. When we integrate, we're going to integrate r from the stack of D3-branes down in the throat (effectively

$r = 0$) up to the position of the brane within the throat, $r \sim \phi\alpha'$. This yields the four-dimensional effective Planck mass for the D3-brane.

To calculate the Ricci scalar, first note that it will decompose into two terms: $R^{(10)} \to R^{(5)} + R_{X_5}$ where the R_{X_5} term may include terms from r if the internal metric is dependent upon r. The only term we care about here, however, is $R^{(5)}$. So, let us calculate $R^{(5)}$ for the metric

$$ds^2 = \frac{r^2}{R^2} g^{(4)}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu + \frac{R^2}{r^2} dr^2. \tag{33}$$

Generally speaking, this is a bit of a pain to do. There's a coordinate transformation and a conformal transformation trick that we can use however. Begin with the coordinate transformation $y = R^2/r$. The metric transforms to

$$ds^2 = \frac{R^2}{y^2} \left[g^{(4)}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu + dy^2 \right]. \tag{34}$$

Now, we can calculate the Ricci scalar of the metric in the brackets (call this metric $g^{(5)}$), and then conformally transform it to find the Ricci scalar for the full metric. Given that the y dimension is flat, the Ricci scalar of the metric in the brackets is just $R^{(4)}[g^{(4)}]$. So, we have

$$R^{(5)}[g^{(5)}_{ab}] = R^{(4)}[g^{(4)}_{ab}]. \tag{35}$$

We want to scale back to a five-dimensional curved metric. Let $\gamma^{(5)}_{ab} = R^2/y^2 g^{(5)}_{ab}$. Then,

$$R^{(5)}[\gamma^{(5)}_{ab}] = \frac{y^2}{R^2} R^{(5)}[g^{(5)}_{ab}] + \text{terms involving derivatives of } \frac{y^2}{R^2}. \tag{36}$$

If you want, you can calculate the derivative terms; any book with an appendix on conformal transformations will have the general form in n dimensions (I don't have a book that has the dependence on the number of dimensions with me however; I only know the 4-dimensional result, and we would need the 5-dimensional result, if we cared about those terms, which we don't). Thus, the piece of $R^{(10)}$ in which we are interested is given by

$$R^{(10)} \supset \frac{y^2}{R^2} R^{(4)}[g^{(4)}_{ab}] = \frac{R^2}{r^2} R^{(4)}[g^{(4)}_{ab}]. \tag{37}$$

Now, we go back to our action, and insert the relevant term. Note that $\sqrt{-g} = \sqrt{-g^{(4)}} r^4/R^4 \cdot R/r \cdot \sqrt{h}$ where h is the metric determinant on the compact angular dimensions.

$$\begin{aligned} S &= \int d^{10}x \sqrt{-g^{(4)}} \sqrt{h} \frac{r^3}{R^3} \frac{e^{-2\Phi}}{\alpha'^4} \frac{R^2}{r^2} R^{(4)}[g^{(4)}_{ab}] && (38) \\ &= \int d^4x \int dr \int d^5y \sqrt{-g^{(4)}} \sqrt{h} \frac{r}{R} \frac{e^{-2\Phi}}{\alpha'^4} R^{(4)}[g^{(4)}_{ab}] && (39) \end{aligned}$$

We can integrate over the compact dimensions to obtain a volume factor for those dimensions. We can also integrate over the r dimension, using appropriate limits.

$$\begin{aligned} S &= \int d^4x \sqrt{-g^{(4)}} V_5 \frac{e^{-2\Phi}}{\alpha'^4} R^{(4)}[g^{(4)}_{ab}] \int_0^{\phi\alpha'} dr \frac{r}{R} && (40) \\ &= \int d^4x \sqrt{-g^{(4)}} \frac{1}{2} \frac{V_0 \phi_{UV}^2}{R\alpha'^2} \frac{\phi^2 V_5 e^{-2\Phi}}{\phi_{UV}^2 V_0} R^{(4)}[g^{(4)}_{ab}] && (41) \end{aligned}$$

Here, the first fraction should be identified as the four-dimensional effective Planck mass m_p^2, while the second fraction is the coupling constant in the string frame. Thus, we have

$$m_p = \frac{\phi_{UV}}{\alpha'} \sqrt{\frac{V_0}{R}}. \tag{42}$$

The Lyth bound is usually given as a bound on the total field variation in inflation, in terms of the tensor to scalar ratio r.

$$\Delta\phi \gtrsim m_P\sqrt{\frac{r}{4\pi}} \tag{43}$$

Inverting this,

$$r \lesssim 4\pi\left(\frac{\Delta\phi}{m_P}\right)^2. \tag{44}$$

Putting in our expression for m_P, we have

$$r \lesssim 4\pi\frac{(\Delta\phi)^2}{\phi_{UV}^2}\frac{R\alpha'^2}{V_0}. \tag{45}$$

Writing this in terms of Δr (the change in the coordinate r) and r_{UV}, this becomes

$$r \lesssim 4\pi\left(\frac{\Delta r}{r_{UV}}\right)^2\frac{R\alpha'^2}{V_0} < 4\pi\frac{R\alpha'^2}{V_0} \tag{46}$$

where the final inequality comes from assuming the brane moves the full range of r_{UV}.