Problem Solution for Week One

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Problem 1.1

(a)

The action for the massless scalar field looks like

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \int d^4x \frac{1}{\eta^4} \eta^2 \left[-(\partial_\eta \phi)^2 + |\nabla_x \phi|^2 \right] \tag{1}$$

From this action the equation of motion can be found to be

$$\partial_\eta^2 \phi - \frac{2}{\eta} \partial_\eta \phi - \nabla^2 \phi = 0 \tag{2}$$

If we change into Fourier space then the equation looks like

$$\partial_\eta^2 \phi - \frac{2}{\eta} \partial_\eta \phi + |k|^2 \phi = 0 \tag{3}$$

If we plug in the trial solution $f(\eta) = C(1 + i|k|\eta)e^{i|k|\eta}$ then we can straight-forwardly calculate the derivatives

$$\partial_\eta f(\eta) = C |k|^2 \eta e^{-i|k|\eta}, \qquad \partial_\eta^2 f(\eta) = C|k|^2 (1 - i|k|\eta) e^{-i|k|\eta} \tag{4}$$

Therefore we have $\partial_\eta^2 f + |k|^2 f = 2C|k|^2 e^{-i|k|\eta} = 2\eta^{-1}\partial_\eta f$. Because the equation is real we expect that f^* will be a solution of the equation as well.

(b)

We know the commutator between creation and annihilation operators $[a, a^\dagger] = 1$. Writing

$$\phi_k(\eta) = f(\eta) a_k^\dagger + f^*(\eta) a_{-k} \tag{5}$$

We can work out the form of the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_\eta \phi)} = -\frac{1}{\eta^2}\partial_\eta \phi = -\frac{1}{\eta^2}\left(f'(\eta) a^\dagger + f'(\eta)^* a \right) \tag{6}$$

Therefore we can just calculate the canonical commutation relation and require it to be i in the end.

$$[\phi, \pi] = -\frac{1}{\eta^2}\left(f^* f' - f f'^* \right) = 2i|C|^2 k^3 \tag{7}$$

Therefore the correct normalization is $C = 1/\sqrt{2k^3}$.

(c)

The Bunch-Davies vacuum is defined through $a\,|BD\rangle = 0$. This definition makes sense because this is the state where there is no quantum excitations. Furthermore, this state is well-defined because if we evolve the time back into $\eta = -\infty$ the theory looks like a free theory in Minkowski space, and there the vacuum is well-defined.

We want to evaluate the quantity $\langle BD\,|\phi_{\mathbf{k}}(\eta)\phi_{-\mathbf{k}}(\eta')|\,BD\rangle$. Plugging in the expression for the Fourier modes of ϕ as in (5), we can get

$$\left\langle BD\left|\left(f(\eta)a_k^\dagger + f^*(\eta)a_{-k}\right)\left(f(\eta')a_{-k}^\dagger + f^*(\eta')a_k\right)\right|BD\right\rangle \tag{8}$$

Only one term survives because by definition we have $a\,|BD\rangle = 0$, so the two point function is

$$\left\langle BD\left|\phi_{\mathbf{k}}(\eta)\phi_{-\mathbf{k}}(\eta')\right|BD\right\rangle = f^*(\eta)f(\eta')\langle BD|BD\rangle = \frac{1}{2\,|k|^3}(1 - i\,|k|\,\eta)(1 + i\,|k|\,\eta')e^{i|k|(\eta-\eta')} \tag{9}$$

(d)

The sum of all the Fourier modes is just the integral over the momentum space

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\mathbf{k}\cdot\Delta\mathbf{x}}}{2\,|k|^3}\left(1 - i\,|k|\,(\eta-\eta') + |k|^2\,\eta\eta'\right)e^{i|k|(\eta-\eta')} = \frac{1}{4\pi^2}\int_0^\infty \frac{dk}{k}\left(1 - ik(\eta-\eta') + k^2\eta\eta'\right)e^{ik(\eta-\eta')}\frac{\sin k\,|\Delta\mathbf{x}|}{k\,|\Delta\mathbf{x}|} \tag{10}$$

The integration can be carried out term by term. The first term gives a logarithmic divergence whereas the other two terms gives a finite contribution. The result of integration is

$$\log(0) - \frac{\eta-\eta'}{|\mathbf{x}-\mathbf{x}'|}\tanh^{-1}\left(\frac{|\mathbf{x}-\mathbf{x}'|}{\eta-\eta'}\right) + \frac{\eta\eta'}{(\eta-\eta')^2 - |\mathbf{x}-\mathbf{x}'|^2} \tag{11}$$

The logarithmic divergence is due to the spectrum of massless field. Because the scalar is massless and the $k = 0$ mode is summed over, and they are extended over the whole space, therefore an infrared divergence is expected in the correlation function.

Problem 2.6

We calculate the 3-point function $\left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle$ of a massless self-interacting scalar field ϕ in de Sitter space. The interaction Hamiltonian H_{int} is given as

$$H_{\text{int}}(\eta) = \int d^3x\,\sqrt{-g}\,\mathcal{H}_{\text{int}}(\eta,\vec{x}) = \int d^3x\sqrt{-g}\frac{M}{6}\phi^3(\eta,\vec{x})\,. \tag{12}$$

g denotes the determinant of the de Sitter metric,

$$ds^2 = \frac{1}{\eta^2}\left(-d\eta^2 + d\vec{x}^2\right)\,,\quad g = -\frac{1}{\eta^8}\,. \tag{13}$$

The interaction Hamiltonian is used to construct the time evolution operator U,

$$U(\eta) = T\exp[-i\int_{-\infty}^{\eta} d\eta' H_{\text{int}}(\eta')]\,. \tag{14}$$

U relates the vacuum $|\Omega\rangle$ of the interacting theory at time η to the Bunch-Davies vacuum $|\mathrm{BD}\rangle$ in the asymptotic past,

$$|\Omega(\eta)\rangle = U(\eta)\,|\mathrm{BD}\rangle\ . \tag{15}$$

The 3-point correlation function can hence be written as

$$\left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle = \langle \Omega(\eta)|\,\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,|\Omega(\eta)\rangle \tag{16}$$

$$= \langle BD|\,U^{-1}(\eta)\,\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,U(\eta)\,|\mathrm{BD}\rangle\ . \tag{17}$$

Expanding U up to first order in the coupling scale M yields

$$\left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle = -i\int_{-\infty}^{\eta} d\eta'\,\langle BD|\left[\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,,H_{\mathrm{int}}\left(\eta'\right)\right]|\mathrm{BD}\rangle\ . \tag{18}$$

Notice that $\langle BD|\,\phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\,|\mathrm{BD}\rangle$ vanishes as an odd number of fields cannot facilitate an vacuum-to-vacuum transition. Next, we insert H_{int} and expand all fields $\phi(\eta',\vec{x})$ into Fourier modes.

$$\left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle = -\frac{iM}{6}\int_{-\infty}^{\eta}\frac{d\eta'}{\eta'^4}\int d^3x\int d^3k_4\int d^3k_5\int d^3k_6\,e^{i\left(\vec{k}_4+\vec{k}_5+\vec{k}_6\right)\vec{x}} \tag{19}$$
$$\times\,\langle BD|\left[\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,,\phi_{\vec{k}_4}\left(\eta'\right)\phi_{\vec{k}_5}\left(\eta'\right)\phi_{\vec{k}_6}\left(\eta'\right)\right]|\mathrm{BD}\rangle\ .$$

The integral over position space provides us with a momentum delta function,

$$\int d^3x\,e^{i\left(\vec{k}_4+\vec{k}_5+\vec{k}_6\right)\vec{x}} = (2\pi)^3\,\delta^{(3)}\left(\vec{k}_4+\vec{k}_5+\vec{k}_6\right)\ . \tag{20}$$

The commutator matrix element in Eq. (19) may be written as

$$\langle BD|\left[\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,,\phi_{\vec{k}_4}\left(\eta'\right)\phi_{\vec{k}_5}\left(\eta'\right)\phi_{\vec{k}_6}\left(\eta'\right)\right]|\mathrm{BD}\rangle \tag{21}$$
$$= \langle BD|\,T\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,,\phi_{\vec{k}_4}\left(\eta'\right)\phi_{\vec{k}_5}\left(\eta'\right)\phi_{\vec{k}_6}\left(\eta'\right)|\mathrm{BD}\rangle - \mathrm{c.c.}$$

To see this, recall that the quantum Fourier modes $\phi_{\vec{k}}$ of the scalar field are related to the solutions $f_{\vec{k}}$ of the classical equations of motion as follows,

$$\phi_{\vec{k}}(\eta) = f_{\vec{k}}(\eta)\,a^\dagger_{\vec{k}} + f^*_{\vec{k}}(\eta)\,a_{\vec{k}} \tag{22}$$

The $\phi_{\vec{k}}$ are thus hermitian such that

$$\langle BD|\,T\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta)\,,\phi_{\vec{k}_4}\left(\eta'\right)\phi_{\vec{k}_5}\left(\eta'\right)\phi_{\vec{k}_6}\left(\eta'\right)|\mathrm{BD}\rangle^* \tag{23}$$
$$= \langle BD|\,\phi_{\vec{k}_6}(\eta)\,\phi_{\vec{k}_5}(\eta)\,\phi_{\vec{k}_4}(\eta)\,,\phi_{\vec{k}_3}\left(\eta'\right)\phi_{\vec{k}_2}\left(\eta'\right)\phi_{\vec{k}_1}\left(\eta'\right)|\mathrm{BD}\rangle$$

The classical equations of motion are solved by

$$f_{\vec{k}}(\eta) = \frac{1}{\sqrt{2k^3}}\,(1+ik\eta)\,e^{-ik\eta}\,,\quad k = \left|\vec{k}\right|\ . \tag{24}$$

The commutator of two Fourier modes therefore turns out to be

$$\begin{aligned}\left[\phi_{\vec{k}}(\eta), \phi_{\vec{k}'}(\eta')\right] &= \left(f^*_{\vec{k}}(\eta) f_{\vec{k}'}(\eta') - f_{\vec{k}}(\eta) f^*_{\vec{k}'}(\eta')\right)\delta^{(3)}\left(\vec{k}-\vec{k}'\right) \\ &= \frac{i}{k^3}\,\mathrm{Im}\Big\{(1-ik\eta)\left(1+ik\eta'\right)e^{-ik(\eta'-\eta)}\Big\}\delta^{(3)}\left(\vec{k}-\vec{k}'\right).\end{aligned} \tag{25}$$

It vanishes for equal times, $\eta' = \eta$, so that the second line of Eq. (23) can be brought into the following form

$$\begin{aligned}&\langle BD|\,\phi_{\vec{k}_6}(\eta)\,\phi_{\vec{k}_5}(\eta)\,\phi_{\vec{k}_4}(\eta), \phi_{\vec{k}_3}(\eta')\,\phi_{\vec{k}_2}(\eta')\,\phi_{\vec{k}_1}(\eta')\,|\mathrm{BD}\rangle \\ =\;& \langle BD|\,\phi_{\vec{k}_4}(\eta)\,\phi_{\vec{k}_5}(\eta)\,\phi_{\vec{k}_6}(\eta), \phi_{\vec{k}_1}(\eta')\,\phi_{\vec{k}_2}(\eta')\,\phi_{\vec{k}_3}(\eta')\,|\mathrm{BD}\rangle\,.\end{aligned} \tag{26}$$

Taken together, Eqs. (23) and (26) prove our statement in Eq. (21).

To evaluate the expectation value of the time-ordered product in Eq. (21) we consult Wick's theorem,

$$\begin{aligned}&\langle BD|\,T\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta), \phi_{\vec{k}_4}(\eta')\,\phi_{\vec{k}_5}(\eta')\,\phi_{\vec{k}_6}(\eta')\,|\mathrm{BD}\rangle \\ =\;& 3\times W\left\{\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\right\} W\left\{\phi_{\vec{k}_3}(\eta)\,\phi_{\vec{k}_4}(\eta')\right\} W\left\{\phi_{\vec{k}_5}(\eta')\,\phi_{\vec{k}_6}(\eta')\right\} + \text{perm.} \\ &+ 6\times W\left\{\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_4}(\eta')\right\} W\left\{\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_5}(\eta')\right\} W\left\{\phi_{\vec{k}_3}(\eta)\,\phi_{\vec{k}_6}(\eta')\right\}.\end{aligned} \tag{27}$$

In the second line cyclic as well as anti-cyclic permutations have to be included. The $W\{\phi\phi\}$ products denote Wick contractions,

$$W\left\{\phi_{\vec{k}}(\eta)\,\phi_{\vec{k}'}(\eta')\right\} = \begin{cases}\left[\phi^+_{\vec{k}}(\eta), \phi^-_{\vec{k}'}(\eta')\right] & ; \quad \eta \geq \eta' \\ \left[\phi^+_{\vec{k}'}(\eta'), \phi^-_{\vec{k}}(\eta)\right] & ; \quad \eta \leq \eta'\end{cases} \tag{28}$$

where

$$\phi^+_{\vec{k}}(\eta) = f^*_{\vec{k}}(\eta)\,a_{\vec{k}}\,, \quad \phi^-_{\vec{k}}(\eta) = f_{\vec{k}}(\eta)\,a^\dagger_{\vec{k}}\,. \tag{29}$$

Analogously to Eq. (25) we find

$$W\left\{\phi_{\vec{k}}(\eta)\,\phi_{\vec{k}'}(\eta')\right\} = \begin{cases} f^*_{\vec{k}}(\eta)\,f_{\vec{k}'}(\eta')\,\delta^{(3)}\left(\vec{k}-\vec{k}'\right) & ; \quad \eta \geq \eta' \\ f_{\vec{k}}(\eta)\,f^*_{\vec{k}'}(\eta')\,\delta^{(3)}\left(\vec{k}-\vec{k}'\right) & ; \quad \eta \leq \eta'\end{cases}. \tag{30}$$

The term in the second line of Eq. (27) is therefore proportional to the following product of delta functions

$$W\left\{\phi_{\vec{k}_1}\phi_{\vec{k}_2}\right\} W\left\{\phi_{\vec{k}_3}\phi_{\vec{k}_4}\right\} W\left\{\phi_{\vec{k}_5}\phi_{\vec{k}_6}\right\} \propto \delta^{(3)}\left(\vec{k}_1-\vec{k}_2\right)\delta^{(3)}\left(\vec{k}_3-\vec{k}_4\right)\delta^{(3)}\left(\vec{k}_5-\vec{k}_6\right) \tag{31}$$

The momenta in the first delta function are not integrated over. For $\vec{k}_1 \neq \vec{k}_2$ this term thus vanishes. Similar arguments apply to all its permutations. Finally, we obtain for the expectation value of the time-ordered product

$$\begin{aligned}&\langle BD|\,T\phi_{\vec{k}_1}(\eta)\,\phi_{\vec{k}_2}(\eta)\,\phi_{\vec{k}_3}(\eta), \phi_{\vec{k}_4}(\eta')\,\phi_{\vec{k}_5}(\eta')\,\phi_{\vec{k}_6}(\eta')\,|\mathrm{BD}\rangle \\ =\;& 6f^*_{\vec{k}_1}(\eta)\,f^*_{\vec{k}_2}(\eta)\,f^*_{\vec{k}_3}(\eta)\,f_{\vec{k}_4}(\eta')\,f_{\vec{k}_5}(\eta')\,f_{\vec{k}_6}(\eta')\,\delta^{(3)}\left(\vec{k}_1-\vec{k}_4\right)\delta^{(3)}\left(\vec{k}_2-\vec{k}_5\right)\delta^{(3)}\left(\vec{k}_3-\vec{k}_6\right)\end{aligned} \tag{32}$$

These three delta functions cancel the momentum integrals in our expression for the 3-point correlation function in Eq. (19),

$$\begin{aligned}\left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle &= -\frac{iM}{6}\left(2\pi\right)^3\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)6\left(f^*_{\vec{k}_1}\left(\eta\right)f^*_{\vec{k}_2}\left(\eta\right)f^*_{\vec{k}_3}\left(\eta\right)I_\eta - \text{c.c.}\right) \\ &= 2M\left(2\pi\right)^3\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right)\text{Im}\left\{f^*_{\vec{k}_1}\left(\eta\right)f^*_{\vec{k}_2}\left(\eta\right)f^*_{\vec{k}_3}\left(\eta\right)I_\eta\right\}\end{aligned} \tag{33}$$

where I_η denotes the remaining time integral,

$$\begin{aligned}I_\eta &= \int\limits_{-\infty}^{\eta}\frac{d\eta'}{\eta'^4}f_{\vec{k}_1}\left(\eta'\right)f_{\vec{k}_2}\left(\eta'\right)f_{\vec{k}_3}\left(\eta'\right) \\ &= \int\limits_{-\infty}^{\eta}\frac{\eta'}{\eta'^4}\frac{\left(1+ik_1\eta\right)}{\sqrt{2k_1^3}}\frac{\left(1+ik_2\eta\right)}{\sqrt{2k_2^3}}\frac{\left(1+ik_3\eta\right)}{\sqrt{2k_3^3}}e^{-ik\eta}\,, \qquad k=k_1+k_2+k_3\,.\end{aligned} \tag{34}$$

It can conveniently be decomposed as follows,

$$\sqrt{8k_1^3k_2^3k_3^3}\,I_\eta = I_\eta^{(4)} + ikI_\eta^{(3)} - \left(k_1k_2+k_1k_3+k_2k_3\right)I_\eta^{(2)} - ik_1k_2k_3I_\eta^{(1)} \tag{35}$$

with $I_\eta^{(n)}$ being defined as

$$I_\eta^{(n)} = \int\limits_{-\infty}^{\eta}\frac{d\eta'}{\eta'^n}e^{-ik\eta}\,. \tag{36}$$

These integrals can be regularized by choosing the integration contour in the complex η plane such that the oscillatory integrad vanishes in the asymptotic past. We set $\eta\to\eta\left(1+i\varepsilon\right)$ with $\varepsilon>0$ which amounts to a smooth transition from the de Sitter vacuum of the interacting theory to the free-field Minkowski vacuum. Integrating by parts provides us with a recursion relation between the integrals $I_\eta^{(n)}$,

$$I_\eta^{(n)} = \frac{1}{\eta'^n}\frac{e^{-i(1+i\varepsilon)k\eta'}}{\left(-i\right)\left(1+i\varepsilon\right)k}\Bigg|_{-\infty}^{\eta} + \frac{n}{\left(-i\right)\left(1+i\varepsilon\right)k}I_\eta^{(n+1)}\,. \tag{37}$$

Taking the limit $\varepsilon\to0$ we get

$$I_\eta^{(n+1)} = \frac{k}{in}I_\eta^{(n)} - \frac{1}{n\eta^n}e^{-ik\eta}\,. \tag{38}$$

With the aid of this relation we can express $I_\eta^{(2)}$, $I_\eta^{(3)}$ and $I_\eta^{(4)}$ in terms of $I_\eta^{(1)}$,

$$I_\eta^{(2)} = -ikI_\eta^{(1)} - \frac{1}{\eta}e^{-ik\eta} \tag{39}$$

$$I_\eta^{(3)} = -\frac{k^2}{2}I_\eta^{(1)} + \left(\frac{ik}{2\eta}-\frac{1}{2\eta^2}\right)e^{-ik\eta} \tag{40}$$

$$I_\eta^{(4)} = i\frac{k^3}{6}I_\eta^{(1)} + \left(\frac{k^2}{6\eta}+\frac{ik}{6\eta^2}-\frac{1}{3\eta^3}\right)e^{-ik\eta} \tag{41}$$

Plugging everything into Eq. (35) gives us

$$\sqrt{8k_1^3k_2^3k_3^3}\, I_\eta = iI_\eta^{(1)}\left[k(k_1k_2+k_1k_3+k_2k_3)-k_1k_2k_3-\frac{k^3}{3}\right] \tag{42}$$
$$+ e^{-ik\eta}\left[\frac{1}{\eta}\left(k_1k_2+k_1k_3+k_2k_3\right)-\frac{k^2}{3\eta}-\frac{ik}{3\eta^2}-\frac{1}{3\eta^3}\right] .$$
$$= -\frac{i}{3}I_\eta^{(1)}\left(k_1^3+k_2^3+k_3^3\right) \tag{43}$$
$$+ e^{-ik\eta}\left[\frac{1}{\eta}\left(k_1k_2+k_1k_3+k_2k_3\right)-\frac{k^2}{3\eta}-\frac{ik}{3\eta^2}-\frac{1}{3\eta^3}\right]$$

This integral now needs to multipled by $f^*_{\vec{k}_1}f^*_{\vec{k}_2}f^*_{\vec{k}_3}$,

$$8k_1^3k_2^3k_3^3\, f^*_{\vec{k}_1}(\eta)\, f^*_{\vec{k}_2}(\eta)\, f^*_{\vec{k}_3}(\eta)\, I_\eta = \left(1-ik\eta-\eta^2\left(k_1k_2+k_1k_3+k_2k_3\right)+i\eta^3k_1k_2k_3\right) \tag{44}$$
$$\times\left[-\frac{i}{3}e^{ik\eta}I_\eta^{(1)}\left(k_1^3+k_2^3+k_3^3\right)+\left[\frac{1}{\eta}\left(k_1k_2+k_1k_3+k_2k_3\right)-\frac{k^2}{3\eta}-\frac{ik}{3\eta^2}-\frac{1}{3\eta^3}\right]\right] .$$

The imaginary part of this expression reads

$$8k_1^3k_2^3k_3^3\,\mathrm{Im}\left\{f^*_{\vec{k}_1}(\eta)\, f^*_{\vec{k}_2}(\eta)\, f^*_{\vec{k}_3}(\eta)\, I_\eta\right\} \tag{45}$$
$$= \left[1-\eta^2\left(k_1k_2+k_1k_3+k_2k_3\right)\right]\left[-\frac{1}{3}\cos(k\eta)\, I_\eta^{(1)}\left(k_1^3+k_2^3+k_3^3\right)-\frac{k}{3\eta^2}\right]$$
$$+ \left[\eta^3k_1k_2k_3-\eta k\right]\left[\frac{1}{3}\sin(k\eta)\, I_\eta^{(1)}\left(k_1^3+k_2^3+k_3^3\right)+\left[\frac{1}{\eta}\left(k_1k_2+k_1k_3+k_2k_3\right)-\frac{k^2}{3\eta}-\frac{1}{3\eta^3}\right]\right] .$$
$$= P_\eta(k_1,k_2,k_3)\, I_\eta^{(1)}+\frac{k}{3}\left(k^2-2k_1k_2-2k_1k_3-2k_2k_3\right)-\frac{1}{3}k_1k_2k_3$$
$$+ \eta^2k_1k_2k_3\left[\left(k_1k_2+k_1k_2+k_2k_3-\frac{k^2}{3}\right)\right]$$
$$= P_\eta(k_1,k_2,k_3)\, I_\eta^{(1)}+\frac{k}{3}\left(k_1^2+k_2^2+k_3^2\right)-\frac{1}{3}k_1k_2k_3+\eta^2k_1k_2k_3\left[\left(k_1k_2+k_1k_2+k_2k_3-\frac{k^2}{3}\right)\right]$$

where P_η is a time-dependent polynomial in k_1, k_2 and k_3,

$$P_\eta(k_1,k_2,k_3) = \frac{1}{3}\left(k_1^3+k_2^3+k_3^3\right)\Big[-\cos(k\eta)\left[1-\eta^2\left(k_1k_2+k_1k_3+k_2k_3\right)\right] \tag{46}$$
$$+ \sin(k\eta)\left[\eta^3k_1k_2k_3-\eta k\right]\Big]$$

Notice that here we have assumed that the leading term from $I_\eta^{(1)}$ is real. Actually the imaginary part of $I_\eta^{(1)}$ also contributes to the above expression. This does not matter, however, as we are mainly interested in the limit $\eta\to 0$ in any case. For very late times we have

$$I_\eta^{(1)} = \int_{-\infty(1+i\epsilon)}^{\eta}\frac{d\eta'}{\eta'}e^{-ik\eta} = -i\pi+\mathrm{Ei}(-ik\eta) \quad \xrightarrow{\eta\to 0} \quad \gamma+\ln(k\eta) . \tag{47}$$

In this limit Eq.(45) becomes

$$\begin{aligned} &8k_1^3k_2^3k_3^3\,\mathrm{Im}\left\{f^*_{\vec{k}_1}(\eta)\,f^*_{\vec{k}_2}(\eta)\,f^*_{\vec{k}_3}(\eta)\,I_\eta\right\} \\ \xrightarrow{\eta\to 0}\; &-\frac{1}{3}\left(k_1^3+k_2^3+k_3^3\right)\left(\gamma+\ln(k\eta)\right)+\frac{k}{3}\left(k_1^2+k_2^2+k_3^2\right)-\frac{1}{3}k_1k_2k_3 \end{aligned} \tag{48}$$

Our final result for the 3-point correlation function takes the following form

$$\begin{aligned} \left\langle \phi_{\vec{k}_1}\phi_{\vec{k}_2}\phi_{\vec{k}_3}\right\rangle = &\frac{M}{12}(2\pi)^3\delta^{(3)}\left(\vec{k}_1+\vec{k}_2+\vec{k}_3\right) \\ &\times\frac{(k_1+k_2+k_3)\left(k_1^2+k_2^2+k_3^2\right)-k_1k_2k_3-\left(k_1^3+k_2^3+k_3^3\right)\left(\gamma+\ln(k\eta)\right)}{k_1^3k_2^3k_3^3} \end{aligned} \tag{49}$$

Problem 2.7

The broken symmetries are the 3 special conformal transformations. The correation functions of ξ are expectacion values in the Bunch-Davies vacuum. This vacuum does not preserve the special conformal transformations, and this is why we don't have that symmetry in the correlation functions, even though the scalar fields are invariant under the full de-Sitter space.

Problem 3.1

The de-Sitter space is defined using the hyperboloid in $\mathbb{R}^{1,4}$

$$-v^2+w^2+x^2+y^2+z^2=R^2 \tag{50}$$

The flat slicing of de-Sitter space is done by using 45 degree hyperplanes of the form $v+w=f(\tau)$, where $f(\tau)$ is an 1-1 function. It makes sense to use the ansatz $f(\tau)=Re^\tau$. To complement this construction, we can have

$$-v+w=Re^{-\tau}-F \tag{51}$$

where F is an undeterminated function. Plugging this into the equation for de-Sitter space, we get

$$F=\frac{x^2+y^2+z^2}{Re^\tau} \tag{52}$$

Plug the above into the metric for $\mathbb{R}^{1,4}$ we can get

$$\begin{aligned} ds^2 &= -dv^2+dw^2+dx^2+dy^2+dz^2 \\ &= -R^2d\tau^2+(x^2+y^2+z^2)d\tau^2-2xdxd\tau-2ydyd\tau-2zdzd\tau+dx^2+dy^2+dz^2 \end{aligned} \tag{53}$$

Now the only thing we need to do is to rescale x, y, and z to cancel the extra and cross terms. This can be done by redefining $x=f(\tau)\hat{x}$, and similarly for y and z. Then we have

$$dx_i=f'(\tau)\hat{x}_id\tau+f(\tau)d\hat{x}_i,\quad dx_i^2=\left[f'(\tau)\right]^2\hat{x}_i^2d\tau^2+2f(\tau)f'(\tau)\hat{x}_id\tau d\hat{x}_i+f^2(\tau)d\hat{x}_i^2 \tag{54}$$

Pluging this into the above metric, in order to get all the cross terms cancel, we simply need $f'(\tau)=f(\tau)$. Solving this equation we conveniently get $f(\tau)=Re^\tau$. Therefore we can plug this into the metric and in the end we get

$$ds^2=R^2\left(-d\tau^2+e^{2\tau}(d\hat{x}^2+d\hat{y}^2+d\hat{z}^2)\right) \tag{55}$$

The spatial metric is indeed a flat one.

Problem 4.4

We can write down the action for the cosmic string after D3-brane inflation

$$S = -\frac{1}{2\pi g_s \alpha'} \int d\sigma d\tau \sqrt{\det\left(G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)} \tag{56}$$

where the metric can be taken as the metric for $\mathrm{AdS}_5 \times X_5$, restricted to the uncompactified spacetime. Because the metric for the whole space is

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\mathbf{x}^2) + \frac{R^2}{r^2}dr^2 + ds^2_{X_5} \tag{57}$$

the restriction is just the first part with a prefactor r^2/R^2. Plugging this metric into the above action, and evaluate it at the place of brane collision at r_0 we can get

$$\begin{aligned} S_{\text{eff}} &= -\frac{1}{2\pi g_s \alpha'} \int d\sigma d\tau \sqrt{\left(\frac{r_0^2}{R^2}\right)^2} \sqrt{\det \eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu} \\ &= -\frac{1}{2\pi g_s \alpha'} \left(\frac{r_0}{R}\right)^2 \int d\sigma d\tau \sqrt{\det\left(\eta_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu\right)} \end{aligned} \tag{58}$$

Therefore the string tension is modified to be

$$T = \frac{1}{2\pi g_s \alpha'} \frac{r_0^2}{R^2} \tag{59}$$

Problem Solution for Week Two

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Problem 4 (Arkani-Hamed)

The equation of motion for the auxiliary field A is

$$\frac{\partial \mathcal{L}}{\partial A} = \sqrt{-g}\left[F''(A)(R-A) - F'(A) + F'(A)\right] = 0 \tag{1}$$

This equation admits two solutions, one is the trivial one $A = R$. This plugged into the original action will give

$$S = \int d^4x\,\sqrt{-g}\left[F'(R)(R-R) + F(R)\right] = \int d^4x\,\sqrt{-g}F(R) \tag{2}$$

The second solution is $F''(A) = 0$, then the term $F(A) + F'(A)(R-A)$ can be treated as a Taylor expansion of $F(R)$ around A, and terms higher than first order vanish identically. Therefore the action is again reduced to

$$S = \int d^4x\,\sqrt{-g}\left[F(A) + F'(A)(R-A)\right] = \int d^4x\,\sqrt{-g}F(R) \tag{3}$$

In order to decouple the field A from the metric we need to do a conformal transformation $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$ to remove the $F'(A)$ in front of R. Under the conformal transformation the square root of the determinant transforms as

$$\sqrt{-g} \to \Omega^4\sqrt{-g} \tag{4}$$

and the scalar curvature transforms as (c.f. Wald)

$$R \to \Omega^{-2}\left[R - 6g^{\mu\nu}\nabla_\mu\nabla_\nu \ln\Omega - 6g^{\mu\nu}(\nabla_\mu \ln\Omega)(\nabla_\nu \ln\Omega)\right] \tag{5}$$

Therefore it is clear that in order to decouple R with $F'(A)$ we only need to choose $\Omega = (F'(A))^{-1/2}$. In addition, the last term in the above equation looks like a kinetic term, if we define a new field as $\sigma \propto \ln\Omega$. Because $\ln\Omega = -(1/2)\ln F'(A)$, we define

$$\sigma = -\ln F'(A) \tag{6}$$

Then the action is transformed to

$$\begin{aligned} S &\to \int d^4x\,\sqrt{-g}\left[R - \frac{3}{2}\nabla_\mu\nabla^\mu\sigma - \frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \Omega^2 A + \Omega^4 F(A)\right] \\ &= \int d^4x\,\sqrt{-g}\left[R - \frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right] - \frac{3}{2}\int d^4x\,\sqrt{-g}\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\nabla^\mu\sigma) \end{aligned} \tag{7}$$

The last term is the integration of a total derivative and can be discarded, and we define the potential as

$$V(\sigma) = \Omega^2 A - \Omega^4 F(A) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \tag{8}$$

So we have shown that a $F(R)$ theory is equivalent to GR plus a scalar with standard kinetic term.

Problem 1 (Zaldarragia)

The dynamics of the scalar perturbations in the hot plasma before recombination are governed by the continuity equation for the density fluctuations δ and the Euler euqation for the divergence θ of the corresponding velocity field. Working in conformal Newtonian gauge Ma and Bertschinger arrive at the following equations for a single uncoupled fluid [1]:

$$\dot{\delta} = -(1+\omega)\left(\theta - 3\dot{\phi}\right) - 3\frac{\dot{a}}{a}\left(\frac{\delta P}{\delta\rho} - \omega\right)\delta\,, \tag{9}$$

$$\dot{\theta} = -\frac{\dot{a}}{a}(1-3\omega)\theta - \frac{\dot{\omega}}{1+\omega}\theta + \frac{\delta P/\delta\rho}{1+\omega}k^2\delta - k^2\sigma + k^2\psi\,. \tag{10}$$

Eq. (10) does not apply to the photon fluid as it is tightly coupled to the baryon fluid through Thomson and Coulomb scatterings. One way to account for these scatterings is to employ the following effective equation for $\dot{\theta}_\gamma$,

$$\dot{\theta}_\gamma = -R^{-1}\left(\dot{\theta}_b + \frac{\dot{a}}{a}\theta_b - c_s^2 k^2 \delta_b\right) + k^2\left(\frac{1}{4}\delta_\gamma - \sigma_\gamma\right) + \frac{1+R}{R}k^2\psi\,. \tag{11}$$

As shown by Ma and Bertschinger this equation is a good approximation to the exact equation in the tight-coupling limit. Finally, the first-order perturbed Einstein equations yield an equation for the evolution of the metric perturbations ϕ and ψ,

$$k^2\left(\dot{\phi} + \frac{\dot{a}}{a}\psi\right) = 4\pi G a^2 \sum_i (\rho_i + p_i)\,\theta_i\,. \tag{12}$$

In a first step we will rewrite Eqs. (9) through (12) as mode equations in Fourier space using the notation and approximations of Seljak in Ref. [2]. We need to apply the following changes:

- We neglect any form of shear stress as well as the sound speed in the baryon fluid:

$$\sigma,\ \sigma_\gamma,\ c_s^2 = 0\,. \tag{13}$$

- We identify the Newtonian potential ψ with the spatial curvature fluctuations ϕ:

$$\psi = \phi\,. \tag{14}$$

- We replace δ, θ and ϕ by their Fourier components (suppressing any wavenumber index).

- We explicitly write θ as the divergence of the velocity field $\vec{v}$. For any given perturbation mode with wavevector $\vec{k} = k\hat{k}$ gravity sources only velocities in the density perturbations parallel to $\hat{k}$. We thus take $\vec{v}$ to be of the form $\vec{v} = -iv\hat{k}$ which implies

$$\theta = i\vec{k}\cdot\vec{v} = kv\,. \tag{15}$$

- We switch to dimensionless time and momentum variables $x = \tau/\tau_r$ and $\kappa = k\tau_r$. Notice that in contrast to Seljak we stick to Ma and Bertschinger's convention and work in units in which $c = 1$. τ_r denotes the would-be conformal time at recombination if the universe had always been matter-dominated after the Big

Bang. We can relate τ_r to observable parameters through the Friedmann equation corresponding to a purely matter-dominated universe. In conformal coordinates we have

$$\left(\frac{da}{d\tau}\right)^2 = \frac{8\pi G}{3}a^4\rho_m = \frac{8\pi G}{3}a\rho_m^0 = H_0^2\Omega_m a\,, \quad \rho_m^0 = \frac{a^3(\tau)}{a^3(\tau_0)}\rho_m\,, \quad a(\tau_0) = 1\,. \tag{16}$$

Setting $a(0) = 0$, Eq. (16) is uniquely solved by $a(\tau) = H_0^2\Omega_m\tau^2/4$. We thus find

$$\tau_r = \frac{2}{H_0}\left(\frac{a_{\rm rec}}{\Omega_m}\right)^{1/2}\,, \quad a_{\rm rec} \simeq 1/1100\,. \tag{17}$$

From now on dots over time-dependent quantities may always refer to derivatives with respect to x,

$$\dot{f} = \frac{d}{dx}f = \tau_r\frac{d}{d\tau}f\,. \tag{18}$$

- We introduce $\eta = \dot{a}/a = \tau_r da/d\tau/a$ and $y_b = \rho_b/\rho_\gamma$ such that $R = \frac{4}{3}\rho_\gamma/\rho_b = \frac{4}{3}y_b^{-1}$.

After these modifications Eqs. (9) through (12) read:

$$\dot{\delta} = -(1+\omega)\left(\kappa v - 3\dot{\phi}\right) - 3\eta\left(\delta p/\delta\rho - \omega\right)\delta\,, \tag{19}$$

$$\dot{v} = -\eta(1-3\omega)v - \frac{\dot{\omega}}{1+\omega}v + \frac{\delta p/\delta\rho}{1+\omega}\kappa\delta + \kappa\phi\,, \tag{20}$$

$$\dot{v}_\gamma = -\frac{3}{4}y_b\left(\dot{v}_b + \eta v_b\right) + \frac{\kappa}{4}\delta_\gamma + \left(1 + \frac{3}{4}y_b\right)\kappa\phi\,, \tag{21}$$

$$\kappa^2\left(\dot{\phi} + \eta\phi\right) = 4\pi G a^2\tau_r^2\sum_i\left(\rho_i + p_i\right)\kappa v_i\,. \tag{22}$$

In a second step, let us use these equations to obtain the explicit mode equations for cold dark matter, photons and the gravitational potential.

Cold dark matter Cold dark matter is pressureless such that $\omega = \delta p_c/\delta\rho_c = 0$. Eqs. (19) and (20) thus turn into

$$\dot{\delta}_c = -\kappa v_c + 3\dot{\phi}\,, \tag{23}$$

$$\dot{v}_c = -\eta v_c + \kappa\phi\,. \tag{24}$$

Photons The EOS for radiation reads $p = \rho/3$ which entails $\omega = \delta p_c/\delta\rho_c = 1/3$. Eq. (19) then becomes

$$\dot{\delta}_\gamma = -\frac{4}{3}\kappa v_\gamma + 4\dot{\phi}\,. \tag{25}$$

In the tight-coupling limit we approximately have $v_b = v_\gamma$. Eq. (20) can be written as

$$\left(1 + \frac{3}{4}y_b\right)\dot{v}_\gamma = -\frac{3}{4}y_b\eta v_\gamma + \frac{\kappa}{4}\delta_\gamma + \left(1 + \frac{3}{4}y_b\right)\kappa\phi\,, \tag{26}$$

or

$$\dot{v}_\gamma = \frac{-y_b\eta v_\gamma + \kappa\delta_\gamma/3}{4/3 + y_b} + \kappa\phi\,. \tag{27}$$

Gravitational potential The Friedmann equation tells us that

$$\eta^2 = \frac{8\pi G}{3} a^2 \tau_r^2 \left(\rho_\gamma + \rho_m\right) \quad \Leftrightarrow \quad 4\pi G a^2 \tau_r^2 = \frac{3\eta^2}{2} \left(\rho_\gamma + \rho_m\right)^{-1} . \tag{28}$$

Evaluating the sum over all species i in Eq. (22) gives

$$\sum_i \left(\rho_i + p_i\right) v_i = \rho_c v_c + \rho_b v_\gamma + \frac{4}{3} \rho_\gamma v_\gamma \tag{29}$$

Putting Eqs. (22), (28) and (29) together, we obtain

$$\dot{\phi} = -\eta\phi + \frac{3\eta^2}{2\kappa} \frac{\rho_\gamma}{\rho_\gamma + \rho_m} \left[v_\gamma \left(4/3 + \rho_b/\rho_\gamma\right) + v_c \rho_c/\rho_\gamma\right] . \tag{30}$$

Let y denote the scale factor normalized to its value at radiation-matter equality, $y = a/a_{\text{eq}}$. We claim that then

$$\frac{\rho_\gamma}{\rho_\gamma + \rho_m} = \frac{1}{1+y} . \tag{31}$$

To prove this, we first determine a_{eq}. Evolving ρ_γ and ρ_γ from $a = a_{\text{eq}}$ to $a = a\left(\tau_0\right) = 1$ provides us with

$$\rho_\gamma^{\text{eq}} = \frac{1}{a_{\text{eq}}^4} \rho_\gamma^0 = \frac{1}{a_{\text{eq}}^3} \rho_m^0 = \rho_m^{\text{eq}} \quad \Rightarrow \quad a_{\text{eq}} = \frac{\rho_\gamma^0}{\rho_m^0} . \tag{32}$$

From this result we immediately see that

$$\frac{1}{1+y} = \frac{1}{1 + \frac{\rho_m^0}{\rho_\gamma^0} a} = \frac{a^{-4}\rho_\gamma^0}{a^{-4}\rho_\gamma^0 + a^{-3}\rho_m^0} = \frac{\rho_\gamma}{\rho_\gamma + \rho_m} . \tag{33}$$

Likewise, we find

$$\frac{\rho_b}{\rho_\gamma} = \frac{\rho_m - \rho_c}{\rho_\gamma} = \frac{\rho_m}{\rho_\gamma} \left(1 - \frac{\rho_c}{\rho_m}\right) = \frac{a^{-3}\rho_m^0}{a^{-4}\rho_\gamma^0} \left(1 - \frac{\Omega_c}{\Omega_m}\right) = y \left(1 - \frac{\Omega_c}{\Omega_m}\right) . \tag{34}$$

With $y_c = \rho_c/\rho_\gamma = y\Omega_c/\Omega_m$ we finally end up at

$$\dot{\phi} = -\eta\phi + \frac{3\eta^2 \left[v_\gamma \left(4/3 + y - y_c\right) + v_c y_c\right]}{2\left(1+y\right)\kappa} . \tag{35}$$

Eqs. (23), (24), (25), (27) and (35) now represent the final set of mode equations. Before we are able to solve them we need to determine the x depedence of y and η. Being proportional to the scale factor y solves the Friedmann equation

$$\dot{y}^2 = \frac{8\pi G}{3} a_{\text{eq}}^2 \tau_r^2 \rho_m^{\text{eq}} (1+y) = 4\alpha^2 (1+y) \tag{36}$$

with (cf. Eq.(17))

$$\alpha^2 = \frac{8\pi G}{3} a_{\text{eq}}^2 \frac{\tau_r^2}{4} \rho_m^{\text{eq}} = \frac{8\pi G}{3} \frac{\tau_r^2}{4} \frac{\rho_m^0}{a_{\text{eq}}} = \frac{a_{\text{rec}}}{a_{\text{eq}}} . \tag{37}$$

The solution of Eq. (36) is given by

$$y = (\alpha x)^2 + 2\alpha x\,, \tag{38}$$

since then

$$\dot{y} = 2\alpha\left(\alpha x + 1\right)\,, \quad \dot{y}^2 = 4\alpha^2\left((\alpha x)^2 + 2\alpha x + 1\right) = 4\alpha^2\left(y + 1\right)\,. \tag{39}$$

From this the x dependence of η trivially follows,

$$\eta = \dot{a}/a = \dot{y}/y = \frac{2\alpha\left(\alpha x + 1\right)}{(\alpha x)^2 + 2\alpha x}\,. \tag{40}$$

At this stage we are ready to hand the mode equations over to Mathematica. Using the command NDSolve we solve our set of equations for the same initial conditions as Seljak in his paper [2]. As initial x value we choose $x_{\text{min}} = 10^{-3}$. The result is shown in figure 1. The only cosmological parameters the scalar perturbations depend

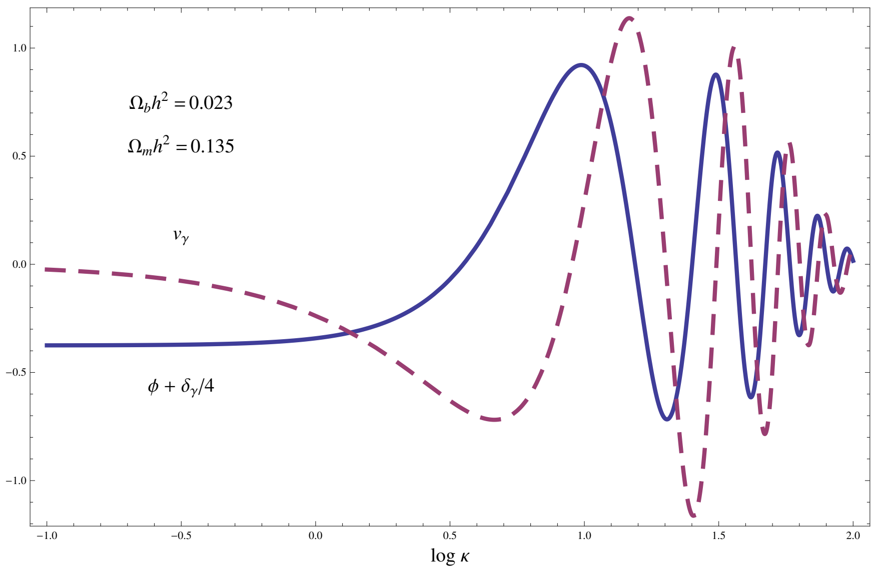

Figure 1: Plot of the Solutions Evolved to Recombination

on are the physical energy densities in matter and baryons $\Omega_m h^2$ and $\Omega_b h^2$. We set this two parameters to the latest WMAP values

$$\Omega_m h^2 = 0.135\,, \quad \Omega_b h^2 = 0.023\,. \tag{41}$$

References

[1] C. P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995) [arXiv:astro-ph/9506072].

[2] U. Seljak, Astrophys. J. **435**, L87 (1994) [arXiv:astro-ph/9406050].