Entropy bounds and the holographic principle

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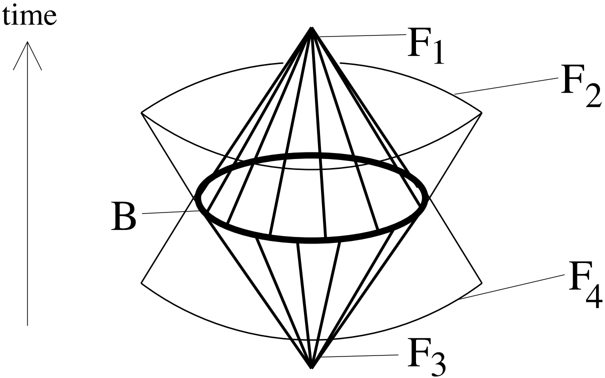
What is the holographic principle?

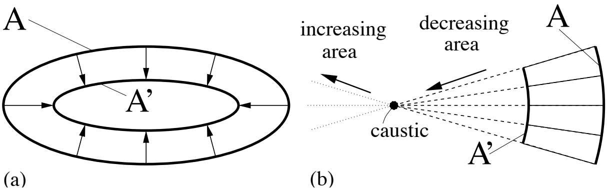
- In its most general form, the holographic principle is a relation between the geometry and information content of spacetime
- This relation manifests itself in the Covariant Entropy Bound

Covariant Entropy Bound



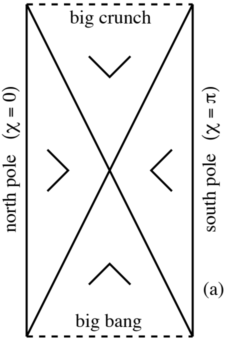
For any two-dimensional surface B of area A, one can construct lightlike hypersurfaces called light-sheets. The total matter entropy on any light-sheet is less than $A/4$ in Planck units: $S \leq A/4G\hbar$.

Covariant Entropy Bound



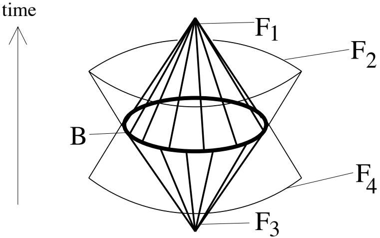
A light-sheet is generated by nonexpanding light-rays orthogonal to the initial surface *B*. Out of the 4 null directions orthogonal to *B*, at least 2 will have this property.

Covariant Entropy Bound



- If B is closed and “normal”, the light-sheet directions will coincide with our intuitive notion of the “interior” of B
- But if B is trapped (anti-trapped) the light-sheets go only to the future (to the past).

What is the holographic principle?



- The CEB is completely general: it appears to hold for arbitary physically realistic matter systems and arbitrary surfaces in any spacetime that solves Einstein's equation
- The CEB can be checked case by case; no counterexamples are known
- But it seems like a conspiracy every time.
  The Origin of the CEB is not known!

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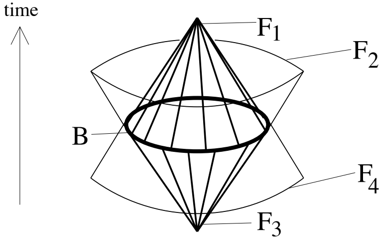

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- This is similar to the “accident” that inertial mass is equal to gravitational charge

What is the holographic principle?

- Solution: Elevate this to a principle and demand a theory in which it could be no other way!
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- In particular, please explain how the CEB and locality fit together!

What is entropy?

- Entropy is the (log of the) number of independent quantum states compatible with some set of macroscopic data (volume, energy, pressure, temperature, etc.)
- The relevant boundary condition for our purposes is that the matter system should fit on a light-sheet of a surface of area A (roughly, that it fits within a sphere of that area)

Plan

- The plan of my lectures is to present the kind of thinking that eventually led to the discovery of the CEB, to explain the CEB in more detail, and to explore its implications
- For a review article, see “The holographic principle”, Reviews of Modern Physics, hep-th/0203101

Black hole entropy

- A black hole is a thermodynamic object endowed with energy, temperature, and entropy [Bekenstein, Hawking, others (1970s)]
- The energy is just the mass; the temperature is proportional to the "surface gravity"; and the entropy is equal to one quarter of the horizon area, in Planck units:

$$S = A/4G\hbar$$

Black hole entropy

- For example, a nonrotating uncharged ("Schwarzschild") black hole of radius R and horizon area $A = 4\pi R^2$ has

$$E = M = R/2G$$

$$S = \pi R^2/G\hbar$$

$$T = \hbar/4\pi R$$

Black hole entropy

- But why do we believe this?
- Black hole entropy was proposed first [Bekenstein 1972], before Hawking discovered black hole temperature and radiation [Hawking 1974]
- Bekenstein's argument went like this:

Do black holes destroy entropy?

- Throw an object with entropy S into a black hole
- By the "no-hair theorem", the final result will be a (larger) black hole, with no classical attributes other than mass, charge, and angular momentum
- This state would appear to have no or negligible entropy, independently of S
- So we have a process in which $dS < 0$
- The Second Law of Thermodynamics appears to be violated!

Bekenstein entropy

- In order to rescue the Second Law, Bekenstein proposed that black holes themselves carry entropy
- Hawking (1971) had already proven the "area law", which states that black hole horizon area never decreases in any process:

$$dA \geq 0$$

- So the horizon area seemed like a natural candidate for black hole entropy. On dimensional grounds, the entropy would have to be of order the horizon area in Planck units:

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- (This was later confirmed, and the factor $1/4$ determined, by Hawking's calculation of the temperature and the relation $dS = dE/T$)

The Generalized Second Law

- With black holes carrying entropy, it is no longer obvious that the total entropy decreases when a matter system is thrown into a black hole
- Bekenstein proposed that a Generalized Second Law of Thermodynamics remains valid in processes involving the loss of matter into black holes
- The GSL states that $dS_{\text{total}} \geq 0$, where

$$S_{\text{total}} = S_{\text{BH}} + S_{\text{matter}}$$

Is the GSL true?

- However, it is not obvious that the GSL actually holds!
- The question is whether the black hole horizon area increases by enough to compensate for the lost matter entropy
- If the initial and final black hole area differ by ΔA, is it true that

$$S_{\text{matter}} \leq \Delta A/4 \ ?$$

- Note that this would have to hold for all types of matter and all ways of converting the matter entropy into black hole entropy!

Testing the GSL

- Let's do a few checks to see if the GSL might be true
- There are two basic processes we can consider:
- Dropping a matter system to an existing black hole, and
- Creating a new black hole by compressing a matter system or adding mass to it

Testing the GSL

- Let's do a few checks to see if the GSL might be true
- There are two basic processes we can consider:
- Dropping a matter system to an existing black hole, and
- Creating a new black hole by compressing a matter system or adding mass to it
- Let's consider an example of the second type

Testing the GSL

- Spherical box of radius R, filled with radiation at temperature at temperature T, which we slowly increase
- Let Z be the effective number of massless particle species
- $S \equiv S_{\text{matter}} \approx ZR^3T^3$, so the entropy increases arbitrarily?!
- However, the box cannot be stable if its mass, $M \approx ZR^3T^4$, exceeds the mass of a black hole of the same radius, $M \approx R$.
- A black hole must form when $T \approx Z^{-1/4}R^{-1/2}$. Just before this point, the matter entropy is

$$S \approx Z^{1/4}A^{3/4}$$

Testing the GSL

- After the black hole forms, the matter entropy is gone and the total entropy is given by the black hole horizon area, $S = A/4$.
- This is indeed larger than the initial entropy, $Z^{1/4}A^{3/4}$, as long as $A \gtrsim Z$, which is just the statement that the black hole is approximately a classical object.
- (We require this in any case since we wish to work in a setting where classical gravity is a good description.)
- So in this example the GSL is satisfied

Entropy bounds from the GSL

- In more realistic examples, such as the formation of black holes by the gravitational collapse of a star, the GSL is upheld with even more room to spare
- As our confidence in the GSL grows, it is tempting to turn the logic around and assume the GSL to be true
- Then we can derive a bound on the entropy of arbitrary matter systems, namely

  $$S_{\text{matter}} \leq \Delta A/4 \ ,$$

  where ΔA is the increase in horizon entropy when the matter system is converted into or added to a black hole

Spherical entropy bound

- For example, consider an arbitrary spherical matter system of mass m that fits within a sphere of area $A \sim R^2$.
- We could presumably collapse a shell of mass $R/G - m$ around this system to convert it into a black hole, also of area A
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- In this sense the world is like a hologram!
- The amount of information needed to fully specify the quantum state in a spherical region fits on its boundary, at a density of order one qubit per Planck area.
- Local QFT is hugely redundant; there are only $\exp(A/4)$ states

Bekenstein bound

- A tighter bound results from a cleverer process:
- Slowly lower the matter system into a very large black hole, to minimize ΔA
- This decreases the energy of the system at infinity, by a redshift factor, before it is dropped in
- The mass added to the black hole is nonzero, however, because the system has finite size
- After some algebra (see hep-th/0203101), one finds

$$S \leq 2\pi MR/\hbar$$

- We will return to this bound later but focus for now on the holographic bound, $S \leq A/4G\hbar$

Limitations

- The derivation of the above bounds from the GSL is somewhat handwaving
- E.g., what if some mass is shed before the black hole forms? It is difficult to treat gravitational collapse processes exactly except in overly idealized limits
- Moreover, the derivation implicitly assumes that we are dealing with a matter system that has weak self-gravity ($M \ll R$).
- Hence, it does not imply that $S \leq A/4$ for all matter systems.
- Will shortly see that indeed, the bound does not hold for some matter systems, if S is naively defined as the entropy "enclosed" by the surface

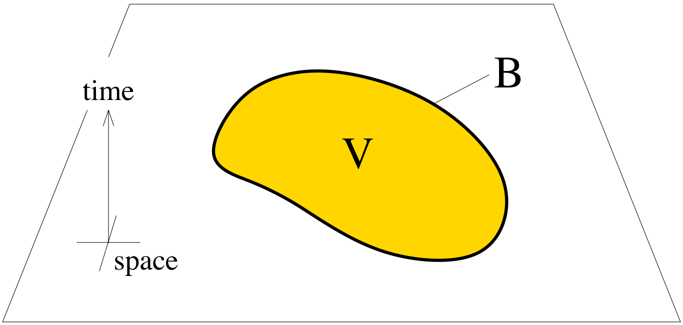
Entropy bounds vs. GSL

- Modern viewpoint: CEB $\to$ GSL.
- CEB is primary and holds for all matter systems
- CEB implies the GSL in the special case where the relevant surface is chosen to lie on the horizon of a black hole
- But CEB holds true in situations where it clearly cannot be derived from the GSL
- CEB reduces to statements resembling the above bounds in certain limits

Towards formulating the CEB

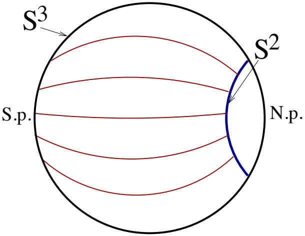
- But how should we define S? Why do we need light-sheets?
- To motivate the CEB, it is instructive to consider a more straightforward guess at a general entropy bound and see why it fails

Spacelike Entropy Bound



- SEB: $S[V] \leq A[B]/4$, for any 3-dimensional volume V
- I will now give four counterexamples to this bound

(1) Closed universe



(a)

- Let V be almost all of a closed three-dimensional space, except for a small region bounded by a tiny sphere B.
- The SEB should apply, $S[V] \leq A[B]/4$, but we can choose $S[V]$ arbitrarily large, and $A[B]$ arbitrarily small

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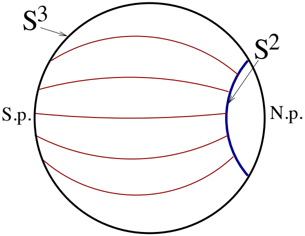

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- The SEB should apply, $S[V] \leq A[B]/4$, but we can choose $S[V]$ arbitrarily large, and $A[B]$ arbitrarily small
- (This type of “arbitrarily bad” violation can be found for any proposed entropy bound other than the CEB; all our counter-examples to the SEB will be of this type.)

(2) Flat FRW universe

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$$

- (E.g., with radiation, $a(t) \sim t^{1/2}$
  and the physical entropy density is $\sigma \sim t^{-3/2}$)
- Consider a volume of physical radius R at fixed time t:

$$V \sim R^3 \ ; A[B] \sim R^2$$

$$S[V] \sim \sigma R^3$$

- In large volumes of space ($R \gtrsim \sigma^{-1}$), the SEB is violated
- $S/A \to \infty$ as $R \to \infty$

(3) Collapsing star

- Consider a collapsing star (idealize as spherical dust cloud)
- Its initial entropy S_0 can be arbitrarily large
- Let V be the volume occupied by the star just before it crunches to a singularity
- (This is well after it crosses its own Schwarzschild radius, so gravity is dominant and the surface of the star is trapped)
- From collapse solutions we know that $A[B] \to 0$ in this limit
- From the (ordinary) Second Law, we know that $S[V] \geq S_0$
- So we can arrange $S[V] > A[B]/4$ and, indeed, $S/A \to \infty$

Give up?

- Perhaps there exists no general entropy bound of the form $S \leq A/4$, which holds for arbitrary regions?
- Instead try to characterize spatial regions that are in some sense sufficiently small, such that the SEB always holds for all of these “special” regions?
- E.g., interior of apparent horizon in FRW, interior of particle horizon, interior of Hubble horizon, etc.?

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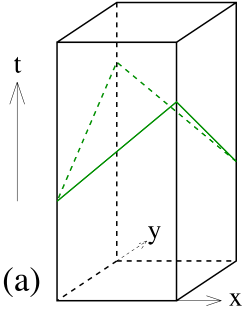
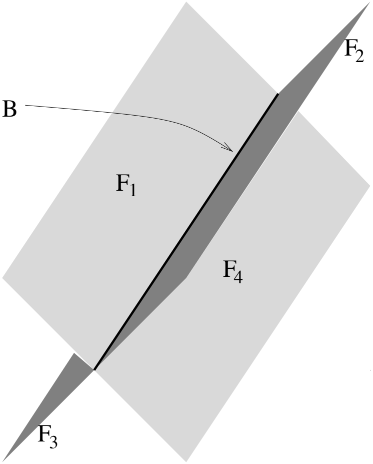
- E.g., interior of apparent horizon in FRW, interior of particle horizon, interior of Hubble horizon, etc.?
- Not well-defined beyond highly symmetric solutions
- Counterexamples have been found to all of these proposals, so
- Retreating from generality doesn’t help!
- The notion of a “sufficiently small spatial region” conflicts with general covariance! (See next counterexample.)

(4) Nearly null boundaries



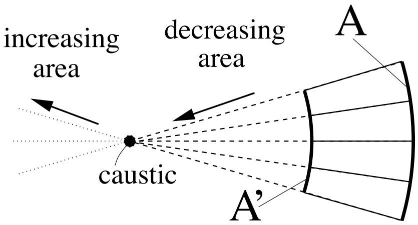
- Consider an ordinary matter system of constant entropy S
- Choose V such that B is Lorentz-contracted everywhere
- In the null limit $A[B] \to 0$, so again,
- the SEB is violated

Null geodesic congruences



- Any 2D spatial surface B bounds four (2+1D) null hypersurfaces
- Each is generated by a congruence of null geodesics $\perp B$

Expansion of a null congruence



- $\theta \equiv \widehat{\nabla_a k^a}$, where k^a is the affine tangent vector field to the congruence (see Wald)
- In terms of an infinitesimal area element $\mathcal{A}$ spanned by nearby light-rays,

$$\theta = \frac{d\mathcal{A}/d\lambda}{\mathcal{A}}$$

- $\theta < 0 \leftrightarrow$ contraction;
- $\theta \to -\infty \leftrightarrow$ caustic (“focal point”, “conjugate point to B”)

Light-sheets

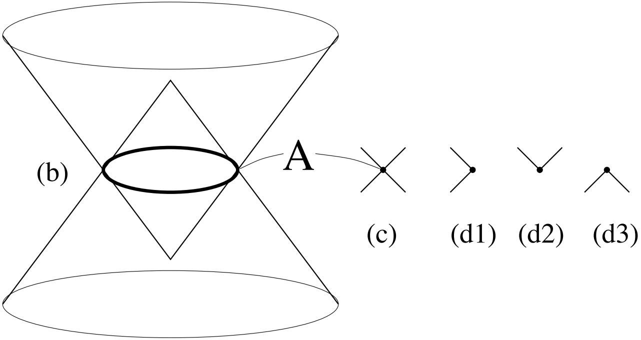
- A light-sheet of B is a null hypersurface $L \perp B$ with boundary B and $\theta \leq 0$ everywhere on L
- Note: Assuming the null energy condition ($T_{ab}k^a k^b \geq 0$) holds,
  - there are at least two null directions away from B for which $\theta \leq 0$ initially
  - $d\theta/d\lambda \leq -\theta^2/2$, so a caustic is reached in finite affine time
- If we think of generating L by following null geodesics away from B, we must stop as soon as θ becomes positive
- In particular, we must stop at any caustic

Covariant Entropy Bound

The total matter entropy on any light-sheet of B is bounded by the area of B:

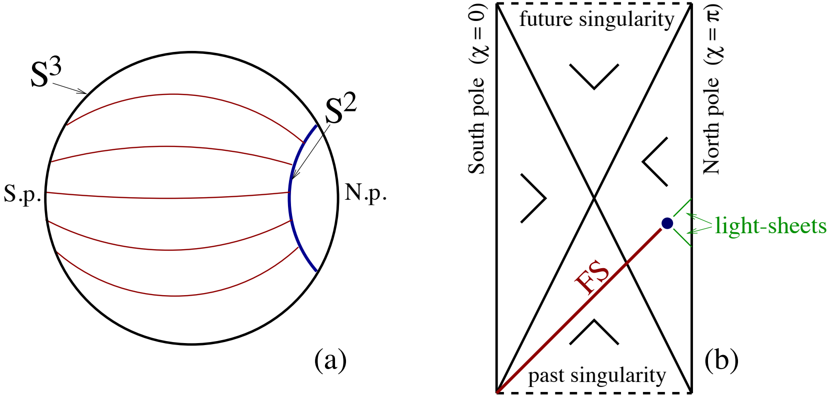
$$S[L(B)] \leq A[B]/4G\hbar$$

Allowed light-sheet directions

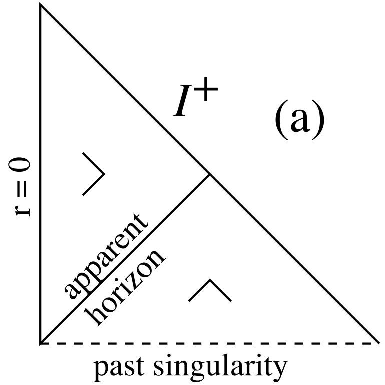


- Often we consider spherically symmetric spacetimes
- In a Penrose diagram, a sphere is represented by a point
- The allowed light-sheet directions can be represented by wedges
- This notation will be useful as we analyze examples

(1) Closed universe

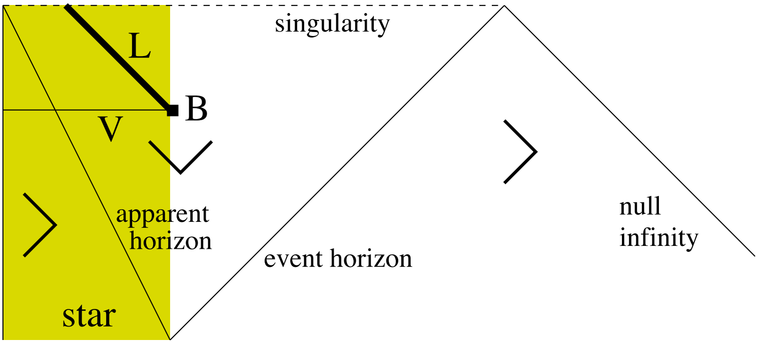


(2) Flat FRW universe



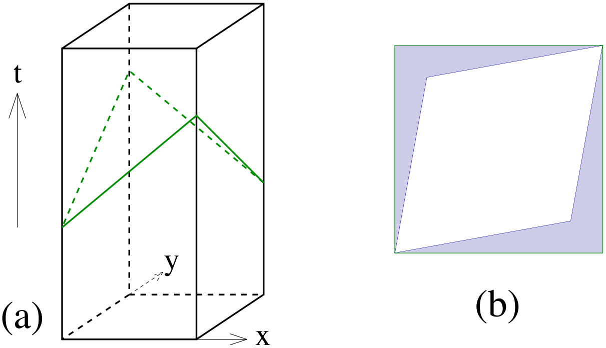
- Sufficiently large spheres at fixed time *t* are anti-trapped
- Only past-directed light-sheets are allowed
- The entropy on these light-sheets grows only like R^2

(3) Collapsing star



- At late times the surface of the star is trapped
- Only future-directed light-sheets exist
- They do not contain all of the star

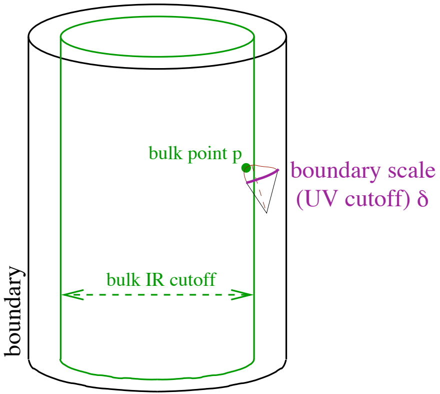
(4) Nearly null boundaries



(a) (b)

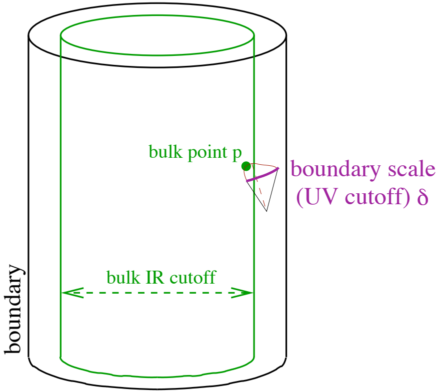
- The null direction orthogonal to B is not towards the center of the system
- The light-sheets miss most of the system, so $S \to 0$ as $A[B] \to 0$

AdS/CFT



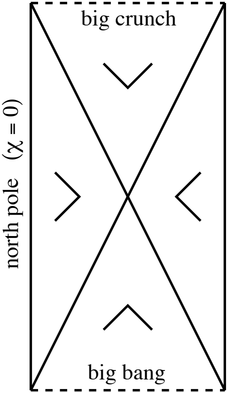
- First complete, nonperturbative quantum theory of gravity
- An asymptotically AdS spacetime is described by a conformal field theory on the boundary

AdS/CFT

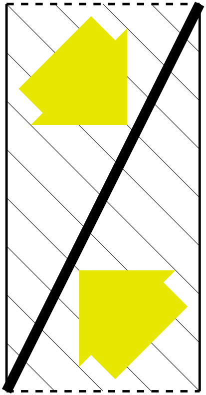


- There exists a cutoff version of this correspondence: A CFT with UV cutoff δ describes AdS out to a sphere of area A
- The relation $A(\delta)$ is such that the log of the dimension of the CFT Hilbert space is of order A
- The holographic principle is manifest!

The world is always a hologram...

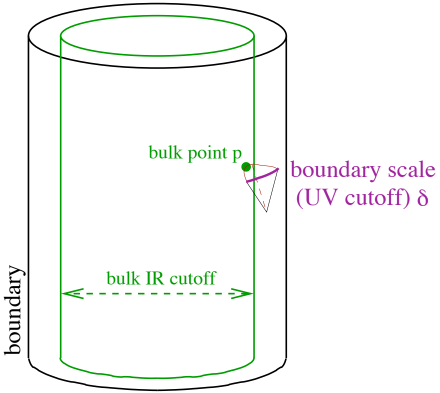


(a)



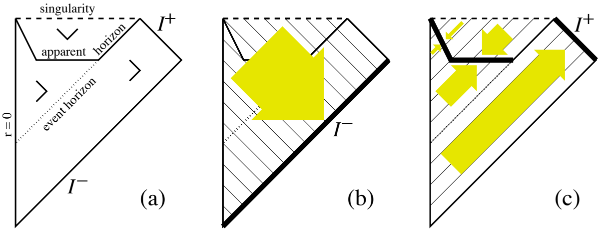
(b)

...but we don't yet know how the encoding works



- AdS is very special; the dual theory is a unitary field theory sharing the same time variable
- This is related to a property of the holographic screen in AdS

Understanding holography in cosmology is hard



- In general spacetimes, it would seem that the number of degrees of freedom has to change as a function of the time parameter along the screen
- The screen is not unique
- The screen can even be spacelike and it need not be connected