1 Problem sets for dS/CFT (Maldacena)

1.1 Review of the two point function computation in dS.

Consider a massless scalar field in dS_4 with metric

$$ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2} \tag{1}$$

a) Write the wave equation and solve it. Show that for each Fourier mode you get the solutions $f \sim (1 + i|k|\eta)e^{-i|k|\eta}$ and its complex conjugate.

b) Write $\phi = a^\dagger f + a f^\dagger$. Demand that $[a, a^\dagger] = 1$, use the canonical commutation relations for ϕ and π_ϕ and normalize f properly.

c) Define the Bunch Davies vacuum through $a|BD\rangle = 0$. Understand why this is a reasonable definition. Then compute

$$\langle BD|\phi_{\vec{k}}(\eta)\phi_{-\vec{k}}(\eta')|BD\rangle$$

d)** After defining this for each fourier mode, could you figure out how it looks once you sum over all Fourier modes?. Can you write the answer in a de Sitter invariant fashion ? Do you get a divergence?, why?

1.2 Interactions

Add a ϕ^3 interaction to the above problem and compute the three point function,

$$\langle BD|\phi_{\vec{k}_1}(\eta)\phi_{\vec{k}_2}(\eta)\phi_{\vec{k}_3}(\eta)|BD\rangle$$

Understand the Keldysh contour necessary for the computation. How do you kill off the oscillatory early time pieces ? Understand that you need to rotate the contour into the imaginary direction.

If you want a simpler problem, then consider a conformally coupled scalar field and add a ϕ^4 interaction. In this case the propagators are simpler and the computation is similar to the flat space one.

1.3 Direct computation of the wavefunction

a) Consider the action for the scalar field $S = \frac{1}{2}\int(\nabla\phi)^2$ in dS. Think about it in the flat slicing.

b) Compute the action with fixed boundary condition at some time η_c, and $\phi_b = \phi(\eta_c)$. At early times, put the positive frequency boundary condition $\phi \to e^{i|k|\eta}$.

c) Evaluate $e^{iS_{classical}}$ for the above solution.

d) Do the same in Euclidean AdS space $ds^2 = \frac{dz^2 + dx^2}{z^2}$. Again put boundary conditions and evaluate the action. Understand the relation to c)

e) Compute the derivative of the wavefunctions with respect to the boundary conditions $\frac{\delta}{\delta\phi_b(\vec{k})}\frac{\delta}{\delta\phi_b(-\vec{k})}$ $e^{iS}|_{\phi_b=0}$. Can this be interpreted as the correlation function of a conformal field theory? Could you go to position space? Are there any IR divergencies in this case?

d) Understand why the Green's function

$$\langle 0|T\phi(\eta)\phi(\eta')|BD\rangle = f^*(\eta)f(\eta') - f(\eta)f(\eta') \ , \quad 0 > \eta > \eta' \tag{2}$$

is the right one to use in the perturbative computation of the wavefunction. You could also try to normalize it correctly.

f)** You could attempt to write this answer in terms of proper distances in AdS. Namely, first go back to position space (don't forget the k dependent normalization factors) and write it in terms of proper distances. Does it have a singularity at the antipodal point?. What is the corresponding function in AdS space, and what is the antipotal point in this case.

1.4 Action for AdS_4

Compute the on shell action for AdS_4 for an S^3 boundary. Compute the action

$$S_{Euclidean} = \frac{R_{AdS}^2}{16\pi G_N} \left[-\int_{\Sigma_4} \sqrt{g}(R+6) - 2\int_{\partial\Sigma_4} K \right] \tag{3}$$

where K is the extrinsic curvature term, $K = \frac{1}{2}h^{ab}\partial_n h_{ab}$ where h is the metric of the boundary and n the normal direction. Write the AdS metric as

$$ds^2 = d\rho^2 + \sinh^2\rho \, d\Omega_3^2 \tag{4}$$

and evaluate the action as a function of the cutoff ρ_c. Discard the divergent terms and focus on the finite ones. Write down the final answer for $\Psi = Z \sim e^{-S_E}$.

1.5 Action for dS_4

Do the same for dS_4 with the Hartle-Hawking analytic continuation. Consider the metric

$$ds^2 = -d\tau^2 + \cosh^2\tau \, d\Omega_3^2 \tag{5}$$

Evaluate now $\Psi \sim e^{iS}$. Compare with the answer in the previous problem.

2 Exercises on Inflation (Creminelli)

1. Using symmetry arguments, calculate the tilt of the spectrum of a scalar with small mass, $m^2 \ll H^2$, in a fixed de Sitter background.

2. Using symmetry arguments, show that the n-point function of ζ in Fourier space in a generic model of inflation is of the form

$$\langle \zeta_{\vec{k}_1} \dots \zeta_{\vec{k}_n} \rangle = (2\pi)^3 \delta(\sum \vec{k}_i) F(k_i) , \tag{6}$$

where F is an homogeneous function of the k's of degree $-3(n-1)$.

3. Calculate the equal time 2-point function of a massless scalar in a fixed de Sitter background in real space. What is the physical meaning of the IR divergence?

4. Photons are massless, but they are not produced during inflation. Why?

5. A Goldstone boson, like an axion, with decay constant f_a lives during inflation and therefore gets quantum perturbations. Estimate the corrections you expect to the free field theory calculation from self interactions, i.e. the deviation from gaussianity. What happens if $H > 4\pi f_a$?

6. Consider a massless scalar φ in de Sitter space with an interaction $\frac{M}{6}\varphi^3$. Calculate the 3-point function $\langle \varphi_{\vec{k}_1} \varphi_{\vec{k}_2} \varphi_{\vec{k}_3} \rangle$.

7. de Sitter is maximally symmetric, i.e. it has 10 isometries. In inflation, the correlation functions of ζ are translationally, rotationally and scale invariant. These are 7 symmetries. What happened to the other 3?

8. Calculate (numerically) $f_{\mathrm{NL}}^{\mathrm{loc.}}$ in a model of modulated reheating coming from the non-linear relation $\zeta(\Gamma)$, as a function of Γ/H, where H is the Hubble parameter at the end of inflation.

3 Problems for Aspects of Eternal Inflation (Susskind)
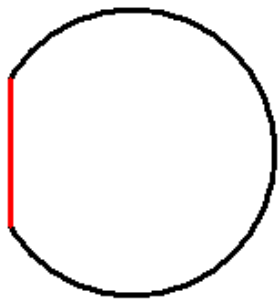

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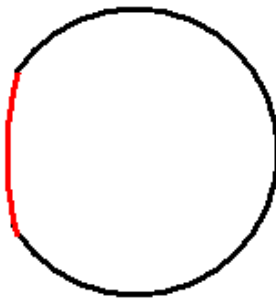
ds to ds

1. Derive the metric of de Sitter space in the flat slicing.

2. Derive the metric of de Sitter space in the static slicing.

3. Prove that the spatial part of the metric in the static slicing corresponds to the metric of a 3 dimensional hemisphere.

4. Write down the rate equations for transitions between vacua. By using detailed balance and redefining variables show that the transition matrix can be made symmetric. Prove that the matrix has a zero eigenvalue and that all others are negative. Find the eigenvector with zero eigenvalue.

5. Consider the case of three vacua labeled $0, 1$ and 2. Assume 0 is terminal. Write down the transition matrix. Show that it has a zero eigenvalue and two negative ones. Find the eigenvector with zero eigenvalue. Show that the probabilities of finding the 1 and 2 vacua decrease with time but that the number of these vacua increases with time. Assume the elements of the transition matrix are small.

6. In the thin wall approximation the CDL instanton is constructed by gluing together two spaces along an infinitely thin seam. For ds $\rightarrow$ flat the spaces are a 4-sphere and a ball of flat space. The size of the ball is a free parameter but is should be less than the radius of the sphere. By analytic continuation find the corresponding Lorentzian signature metric in all regions I, II,

III, IV, IV. For ds → ds the flat plane is replaced by a cap of a larger 4-sphere. Repeat the previous problem for this case. In the process you will discover a 4th slicing of de Sitter. (See figure)

7. Consider a model in which the nucleation event is followed by a period of ordinary slow roll inflation that lasts for N_e efoldings. Carry out the analysis of the measure $M(h, N_e)$. (Keep in mind that the measure counts the average number of observers.) Do it for the proper-time, scale-factor, and light-cone-time cutoffs. (See figure)
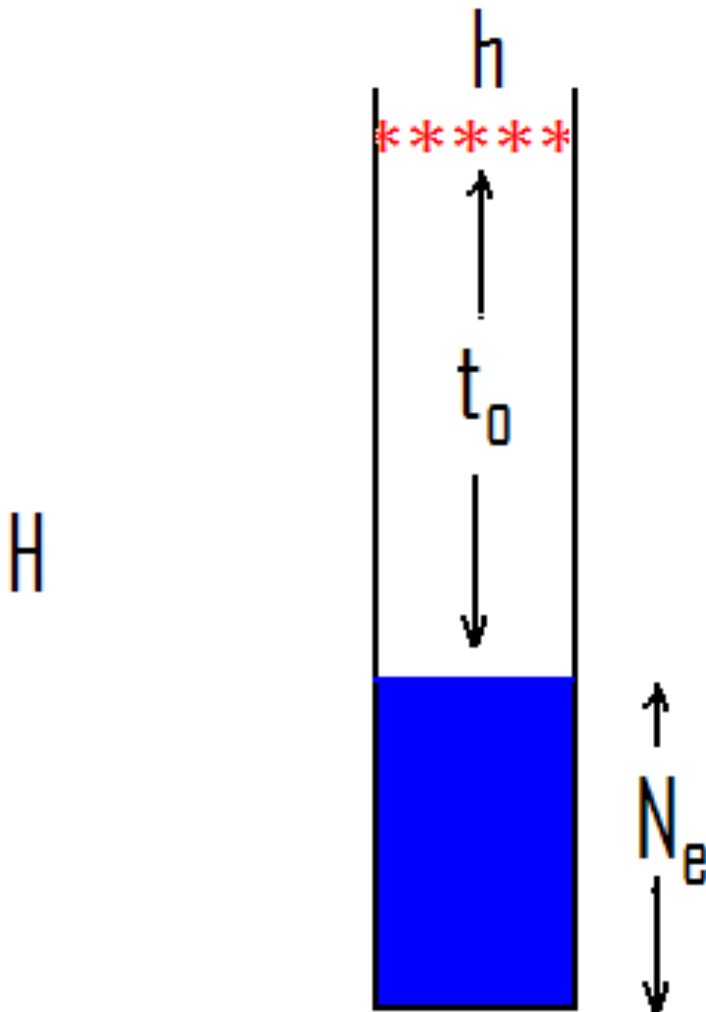
4 Mechanisms for Inflation (E. Silverstein)

1) Consider a model of inflation based on a $\mu^{4-p}\lambda\phi^p$ potential (in the regime $\phi > M_P$), with $0 < p \leq 2$.

a) Determine the conditions on μ and the field range of ϕ during inflation to obtain N_e e-foldings of inflation, with a primordial power spectrum of the correct COBE-normalized magnitude over the

e-foldings visible in the CMB.

b) Analyze the radiative stability of this model. Noting that, as discussed in the lectures, μ can arise as an exponentially small scale, discuss whether or not inflation over $N_e \geq 60$ e-foldings requires fine-tuning of the effective action from the Wilsonian point of view.

c) Discuss additional stability criteria which can arise in a UV completion of inflationary models (using string theory as a concrete example if you wish).

2) In this problem you will explore a few features of string compactifications and scalar fields that descend from them.

a) Consider compactifying a D-dimensional theory of gravity on a $D-4$ dimensional manifold X of linear size R. To be concrete, you can consider X to be a sphere, a torus, or compact hyperbolic space. This leads to a number of scalar fields in four dimensions, one of which corresponds to the volume V_X of X. By dimensionally reducing the Einstein term, show that the corresponding canonically normalized scalar field σ_X is given by

$$V_X = V_0 e^{c_X \sigma_X / M_P}$$

with c_X a number which is not parametrically small. If you feel energetic, include the string theory dilaton Φ and analyze the combined system of Φ and σ_X in the same way.

b) An important ingredient in string theory compactification is the *orientifold*. Without needing too much detail, this object sources a negative gravitational potential, but also introduces a Z_2 identification of space transverse to it (reflecting all the coordinates transverse to its extended directions). Discuss the difference between this and, say, a negative mass particle (or negative mass Schwarzchild solution) in terms of stability of the theory and validity of the second law of thermodynamics. (For the latter part, recall that the area of a black hole behaves like entropy.)

c) In the lectures we consider how axions $b = \int_\Sigma B$ work in string theory, with terms of the form $|dC_p + B \wedge dC_{p-2}|^2$ appearing in the effective action in addition to the kinetic energy $|dB|^2$. The latter term respects a gauge symmetry, $B \to B + d\Lambda_1$, where Λ_1 is a on-form gauge potential. Show that the former term also respects this symmetry, with an appropriate transformation law for C_p.

d) In the lectures we are also discussing brane motion inside compactifications. Consider for example D3-branes in a warped region. Consider an $AdS_5 \times X_5$ throat

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + ds^2_{X_5}$$

cutoff at $r = r_{UV}$ and connected to a compactification. A 3-brane can move in r, with canonically normalized field $\phi \sim r/\alpha'$. Compute the four-dimensional Planck mass as a function of the maximum value of the canonically normalized field $\phi_{UV} \sim r_{UV}/\alpha'$ and the geometry of the compact dimensions. Using the Lyth bound, convert this to a bound on the tensor to scalar ratio as a function of these quantities.

3) Derive the equations of motion for the homogeneous field and perturbations for inflation governed by action $S = -\int d^4x \sqrt{-g}[(\phi^4/\lambda)\sqrt{1 - \lambda(\partial_\mu \phi \partial^\mu \phi)/\phi^4} - V(\phi)]$. Show explicitly that inflation occurs on a steep potential, one which does not satisfy the slow roll conditions.

4) Consider the model of warped D-brane slow-roll inflation discussed in the lectures. Estimate the tension of cosmic strings produced at the end (in suitable cases) as a function of the parameters of the model.

5 CMB & LSS (Zaldarriaga)

1. The objective of this problem is to reproduce the results of Seljak '94 (S94). Starting with the equations in Ma & Bertschinger '95 derive equation (3) in S94. Derive the solutions for the evolution of the background quantities y, η as a function of x. Write a routine in Mathematica (using NDSolve) that solves the equations for the perturbations up to recombination for a given value of κ and cosmological parameters. Reproduce the top panel of figure 1. Derive equation (5) starting from the integral solution. Make a spline of the sources at recombination as a function of κ and integrate them to obtain C_l. Reproduce the bottom panel of figure 2. Include Silk damping as done in S94. For the more ambitious you can include damping by adding shear viscosity directly to the equations (see Mukhanov's book).

2. **Sachs-Wolfe effect in ζ gauge:** We want to consider the temperature differences we see on the sky produced by a long wavelength mode which was outside the horizon at the time of recombination, the so-called Sachs-Wolfe limit. In this limit the recombination must have proceeded unaffected in the sense that an observer at recombination could not have detected the presence of the long mode. Any temperature difference we observe is a sort of projection effect. We want to do the computation in the ζ gauge which should make this fact manifest. We will assume the universe is matter dominated. Follow 0806.1016 and write down the metric in the ζ gauge during the matter era (note you cannot neglect N_i). Use the fact that the observed temperature in a given direction is

$$T_{obs} = T_e \frac{(P^\mu u_\mu)_o}{P^\mu u_\mu)_e} \tag{7}$$

where T_e is the CMB temperature where it was emitted and is unperturbed, P^μ is the momentum of the photon and $u_m u$ is the four velocity of the fluid at the emission and measurement points. Using this equation and the geodesic equation for the photons find $\delta T/T$ in terms of the values of ζ at the last scattering surface. Does your answer agree with the standard answer in the Newtonian gauge?

3. Assuming the photons undergo a random walk, estimate the length they will have diffused by the time of last scattering. Using this length derive the from of the damping used in S94. What angle will this length subtend on the sky?

4. Use the equations derived in problem 1 to determine the parameters of the cosmological model that govern the dynamics of the perturbations up to recombination. What is the effect of changing the the distance to the last scattering surface? Assume you want to determine Ω_m, Ω_b, Ω_Λ and h using the CMB temperature power spectra. Argue that there is a degeneracy between parameters.

5. The objective of this problem is to get a sense of how the temperature power spectrum depends on the cosmological parameters and how well current data can determine these parameters. We will consider the following parameters to describe the matter content of the universe $(\Omega_\Lambda, \Omega_m h^2, \Omega_b h^2)$ and we will restrict ourselves to flat models, $(\Omega_k = 0)$. To describe the power spectrum of initial curvature fluctuations we will use the amplitude A and the spectral index n. We will not consider gravity waves.

We will compare theoretical models with the latest WMAP. All the necessary ingredients can be found in LAMBDA http://lambda.gsfc.nasa.gov/. Use CAMB online tool from LAMBDA to compute the temperature power spectra for the WMAP best fit model. You can choose the specific parameter from the WMAP best fit table also in LAMBDA.

Produce three families of models with one parameter of $(\Omega_\Lambda, \Omega_m h^2, \Omega_b h^2)$ varying in each. Produce plots that show the C_l for each family. In these plots also show the WMAP data with its corresponding error bars (you can download a table with the power spectra and its error bars from LAMBDA).

For each family explain the physics behind the changes in the power spectra. Roughly estimate (by comparing the changes produced by each parameter with the error bars in the data) the range of values for each parameter seems acceptable. Compare with what is given in the table of best fit parameters.

6. BAO transfer function: Estimate the effect on the total matter transfer function of the acoustic oscillations in the photon-baryon that occur prior to recombination. Assume that the decoupling of the baryons happens well inside the matter era. Assume that instantaneously the baryons are released with a density and velocity given by those calculated assuming they were perfectly coupled to the photons prior to the decoupling. After decoupling consider the combined fluid of dark matter and baryons. Use the properly combined density and velocity to obtain the solution for the dark matter + baryon fluid after the decoupling. You will match these initial conditions to a superposition of the growing and decaying modes during the matter era. How big are the resulting features in the power spectrum at late times? What are the k values of the peaks in the spectrum and how do they compare with the k values for the peak in the CMB density fluctuations?

6 Robustness of GR. Attempts to modify gravity (Arkani-Hamed)

1. Compute the "polarization sum" for gravitons in D dimensions, and use it to determine the structure of the graviton propagator.

2. Consider a collection of spin-1 particles with $(--+)$ and $(++-)$ 3-point amplitudes given by f_{abc}. Show that consistent factorization of the 4-point amplitudes forces f_{abc} to satisfy the Jacobi identity.

3. Consider the leading Galilean-invariant interaction of the form $\frac{1}{\Lambda^3}\partial\pi\partial\pi\partial^2\pi$, with a coupling to matter $\frac{1}{M_{Pl}}T\pi$. Show that the fluctuations of π are around a spherically symmetric matter source are superluminal.

4. Let us verify that an action of the form

$$S = \int d^4x\sqrt{-g}F(R) \tag{8}$$

is equivalent to GR + a scalar with standard kinetic term. Show that the action above is equivalent to

$$\int d^4x\sqrt{-g}\left[F'(A)(R-A) + F(A)\right] \tag{9}$$

once the equation of motion for the auxiliary field A is used. Now we can make a conformal transformation to demix A from the metric; this will give a kinetic term to A. After dust has settled you should get

$$\int d^4x\sqrt{-g}\left[R - \frac{3}{2}g^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - V(\sigma)\right] \tag{10}$$

with

$$V(\sigma) = \frac{A}{F'(A)} - \frac{F(A)}{F'(A)^2} \qquad \text{with } \sigma = -\log F'(A) \,. \tag{11}$$

7 Robustness of GR. Attempts to modify gravity (Arkani-Hamed)

1. Determine the form of the bias as a function of mass in the Press-Schechter model.