PiTP Study Guide to Spinors in 11D and 10D

S.J.Gates, Jr.
John S. Toll Professor of Physics
Director, Center for String and Particle Theory
University of Maryland Tel: 301-405-6025
Physics Department Fax: 301-314-9525
Rm. 4125 E-mail: gatess@wam.umd.edu
College Park, MD 20742-4111
http://www.physics.umd.edu/ep/gates/gates.html
http://nsbp.org/cgi-bin/nsbp.cgi?page=jgates

Spinors and σ-matrices in 10 dimensions

It is well known that in D-dimensional space-time Dirac spinor has $2^{\frac{D}{2}}$ complex components if D is even and in that case all the representations of the Dirac algebra are equivalent, while in odd dimensional space-time the number of components is $2^{\frac{D-1}{2}}$ with two inequivalent representations. This result is valid for any signature of the flat space-time metric. However, this kind of spinor is not for any D the smallest irreducible representation. When D is even we can always impose the Weyl condition which separates the spinor into its left and right handed components which transform independently under the Lorentz group. Each of these Weyl spinors thus has only half degrees of freedom of the original Dirac spinor. One more condition can be imposed to reduce further the number of independent components. This is so called Majorana condition which relates the charge-conjugate Dirac spinor to the original one. If after the charge conjugation the spinor is not changed, the number of independent components is reduced again by a factor of two. In that case the left and right Weyl spinors are related. The resulting spinor is necessarily real. This type of reduction can be done in 2 (mod 8) dimensional space-time. It then follows that for $D = 10$ we can impose both conditions and have for the basic spinor a 16-dimensional real object. Consequently, instead of using the Dirac gamma matrices we use the Pauli matrices, of dimension 16×16 with real elements. We can then use small Greek index to denote 16 component left-handed Majorana-Weyl spinor and dotted index for a right-handed one:

$$(\psi^{\alpha})^{*} = (\psi^{\alpha}), \qquad (\chi^{\dot{\alpha}})^{*} = (\chi^{\dot{\alpha}}). \qquad (A.1)$$

The indices can be raised and lowered by the use of the charge conjugation matrix which has mixed indices: $C_{\alpha\dot\beta}$ and $C^{\alpha\dot\beta}$,

$$\begin{aligned} \psi_{\dot\beta} &= \psi^{\alpha} C_{\alpha\dot\beta}, & \chi_{\alpha} &= \chi^{\dot\beta} C_{\alpha\dot\beta}, \\ C_{\alpha\dot\gamma} C^{\alpha\dot\beta} &= \delta^{\dot\beta}_{\dot\gamma}, & C^{\alpha\dot\beta} C_{\gamma\dot\beta} &= \delta^{\alpha}_{\gamma}. \end{aligned} \tag{A.2}$$

As we can see, raising and lowering the indices also changes the handedness of a spinor. The Dirac algebra (of the Pauli matrices) is defined through the usual anti-commutation relation:

$$(\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{b}})^{\beta\gamma} + (\sigma_{\underline{b}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} = -2\eta_{\underline{ab}}\delta^{\gamma}_{\alpha} \ . \tag{A.3}$$

The sigma matrices can be regarded as bispinors. There are three types of them: purely left-handed, purely right-handed and mixed ones. The purely left-handed ones are

$$(\sigma^{\underline{a}})_{\alpha\beta}, \quad (\sigma^{\underline{abc}})_{\alpha\beta}, \quad (\sigma^{\underline{a}_1\underline{a}_2\underline{a}_3\underline{a}_4\underline{a}_5})_{\alpha\beta}, \tag{A.4}$$

where lower case Latin indices are vector indices denoting space-time directions. Purely right-handed bispinors have only dotted indices, but due to the existence of the charge conjugation matrix we have

$$(\sigma^{\underline{a}})^{\alpha\beta} = C^{\alpha\dot\alpha} C^{\beta\dot\beta} (\sigma^{\underline{a}})_{\dot\alpha\dot\beta}. \tag{A.5}$$

The mixed bispinors are

$$C_{\alpha\dot\beta}, \quad (\sigma^{\underline{ab}})_{\alpha\dot\beta}, \quad (\sigma^{\underline{abcd}})_{\alpha\dot\beta}. \tag{A.6}$$

They are, of course, related to

$$\delta_{\alpha}{}^{\beta}, \quad (\sigma^{\underline{ab}})_{\alpha}{}^{\beta}, \quad (\sigma^{\underline{abcd}})_{\alpha}{}^{\beta}. \tag{A.7}$$

The sigma matrices with more vector indices are defined through the following multiplication table:

$$\begin{aligned} (\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{b}})^{\beta\gamma} &= -\eta_{\underline{ab}}\delta_{\alpha}{}^{\gamma} - (\sigma_{\underline{ab}})_{\alpha}{}^{\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bc}})_{\gamma}{}^{\beta} &= -\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{c}]})_{\alpha\gamma} - (\sigma_{\underline{abc}})_{\alpha\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bcd}})^{\beta\gamma} &= -\frac{1}{2!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cd}]})_{\alpha}{}^{\gamma} - (\sigma_{\underline{abcd}})_{\alpha}{}^{\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bcde}})_{\gamma}{}^{\beta} &= \frac{1}{3!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cde}]})_{\alpha\gamma} + (\sigma_{\underline{abcde}})_{\alpha\gamma}, \\ (\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bcdef}})^{\beta\gamma} &= -\frac{1}{4!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cdef}]})_{\alpha}{}^{\gamma} - \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}(\sigma^{[4]})_{\alpha}{}^{\gamma}. \end{aligned} \tag{A.8}$$

$$
\begin{aligned}
(\sigma_{\underline{a}})_{\alpha\beta}(\sigma_{\underline{b}})^{\beta\gamma} &= -\,\eta_{\underline{ab}}\delta_{\alpha}{}^{\gamma} \;-\; (\sigma_{\underline{ab}})_{\alpha}{}^{\gamma}, \\
(\sigma_{\underline{bc}})_{\beta}{}^{\alpha}(\sigma_{\underline{a}})^{\beta\gamma} &= +\,\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{c}]})^{\alpha\gamma} \;-\; (\sigma_{\underline{abc}})^{\alpha\gamma}, \\
(\sigma_{\underline{bcd}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} &= -\,\frac{1}{2!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cd}]})_{\alpha}{}^{\gamma} \;+\; (\sigma_{\underline{abcd}})_{\alpha}{}^{\gamma}, \\
(\sigma_{\underline{bcde}})_{\beta}{}^{\alpha}(\sigma_{\underline{a}})^{\beta\gamma} &= -\,\frac{1}{3!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cde}]})^{\alpha\gamma} \;+\; (\sigma_{\underline{abcde}})^{\alpha\gamma}, \\
(\sigma_{\underline{bcdef}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} &= -\,\frac{1}{4!}\eta_{\underline{a}[\underline{b}}(\sigma_{\underline{cdef}]})_{\alpha}{}^{\gamma} \;+\; \frac{1}{4!}\epsilon_{\underline{abcdef}[4]}(\sigma^{[4]})_{\alpha}{}^{\gamma}.
\end{aligned}
\tag{A.9}
$$

All of the sigma matrices are totally antisymmetric in their vector indices. The sigma matrices with one or five vector indices are symmetric with respect to spinor indices, while the matrix with three vector indices is antisymmetric in spinor indices. The symmetrization and antisymmetrization is denoted in the following way:

$$
\begin{aligned}
A_{(\alpha}B_{\beta)} &= A_{\alpha}B_{\beta} \;+\; A_{\beta}B_{\alpha}, \\
A_{[\alpha}B_{\beta]} &= A_{\alpha}B_{\beta} \;-\; A_{\beta}B_{\alpha}.
\end{aligned}
\tag{A.10}
$$

The sigma matrices with five vector indices satisfy the identities:

$$
\begin{aligned}
(\sigma_{[5]})_{\alpha\beta} &= \frac{1}{5!}\epsilon_{[5][\bar{5}]}(\sigma^{[\bar{5}]})_{\alpha\beta}, \\
(\sigma_{[5]})^{\alpha\beta} &= -\,\frac{1}{5!}\epsilon_{[5][\bar{5}]}(\sigma^{[\bar{5}]})^{\alpha\beta}.
\end{aligned}
\tag{A.11}
$$

In writing these expressions, we have introduced a handy notation for multiple anti-symmetrical indices. So for example,

$$
(\sigma^{[\bar{5}]})_{\alpha\beta} \;\equiv\; (\sigma^{\underline{a}_1\underline{a}_2\bar{\underline{a}}_3\underline{a}_4\underline{a}_5})_{\alpha\beta}
\tag{A.12}
$$

It is also useful to define matrices with more than five indices

$$
\begin{aligned}
(\sigma_{[6]})_{\alpha}{}^{\beta} &= \frac{1}{4!}\epsilon_{[6][\bar{4}]}(\sigma^{[\bar{4}]})_{\alpha}{}^{\beta}, \\
(\sigma_{[7]})_{\alpha\beta} &= -\,\frac{1}{3!}\epsilon_{[7][\bar{3}]}(\sigma^{[\bar{3}]})_{\alpha\beta}, \\
(\sigma_{[7]})^{\alpha\beta} &= -\,\frac{1}{3!}\epsilon_{[7][\bar{3}]}(\sigma^{[\bar{3}]})^{\alpha\beta}, \\
(\sigma_{[8]})_{\alpha}{}^{\beta} &= -\,\frac{1}{2!}\epsilon_{[8][\bar{2}]}(\sigma^{[\bar{2}]})_{\alpha}{}^{\beta}, \\
(\sigma_{[9]})_{\alpha\beta} &= \epsilon_{[9]\underline{a}}(\sigma^{\underline{a}})_{\alpha\beta}, \\
(\sigma_{[9]})^{\alpha\beta} &= \epsilon_{[9]\underline{a}}(\sigma^{\underline{a}})^{\alpha\beta}.
\end{aligned}
\tag{A.13}
$$

The trace identities are present next.

$$
\begin{aligned}
(\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{\underline{c}})^{\alpha\beta} &= -16\ \delta_{\underline{a}}^{\underline{c}}\ , \\
(\sigma_{\underline{ab}})_{\alpha}{}^{\beta}(\sigma^{\underline{cd}})_{\beta}{}^{\alpha} &= -16\ \delta_{[\underline{a}}^{\underline{c}}\delta_{\underline{b}]}^{\underline{d}}\ , \\
(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{def}})^{\alpha\beta} &= -16\ \delta_{[\underline{a}}^{\underline{d}}\delta_{\underline{b}}^{\underline{e}}\delta_{\underline{c}]}^{\underline{f}}\ , \\
(\sigma_{\underline{abcd}})_{\alpha}{}^{\beta}(\sigma^{\underline{efgh}})_{\beta}{}^{\alpha} &= 16\ \delta_{[\underline{a}}^{\underline{e}}\delta_{\underline{b}}^{\underline{f}}\delta_{\underline{c}}^{\underline{g}}\delta_{\underline{d}]}^{\underline{h}}\ , \\
(\sigma_{\underline{abcde}})_{\alpha\beta}(\sigma^{\underline{fghij}})^{\alpha\beta} &= -16\ [\delta_{[\underline{a}}^{\underline{f}}\delta_{\underline{b}}^{\underline{g}}\delta_{\underline{c}}^{\underline{h}}\delta_{\underline{d}}^{\underline{i}}\delta_{\underline{e}]}^{\underline{j}} + \epsilon_{\underline{abcde}}{}^{\underline{fghij}}].
\end{aligned}
\qquad (A.14)
$$

Finally, the following Fierz identities are valid.

1. $(\sigma^{\underline{a}})_{(\alpha\beta}(\sigma_{\underline{a}})_{\gamma)\delta} = 0,$
2. $(\sigma^{\underline{a}})_{\alpha(\beta}(\sigma_{\underline{a}})_{\gamma)\delta} = -(\sigma^{\underline{a}})_{\beta\gamma}(\sigma_{\underline{a}})_{\alpha\delta},$
3. $(\sigma^{\underline{a}})^{\alpha(\beta}(\sigma_{\underline{a}})^{\gamma)\delta} = -(\sigma^{\underline{a}})^{\beta\gamma}(\sigma_{\underline{a}})^{\alpha\delta},$
4. $(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})_{\gamma\delta} = 0,$
5. $(\sigma^{\underline{abc}})_{\alpha\beta}(\sigma_{\underline{abc}})^{\gamma\delta} = -8\cdot 3!\ \delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta},$
6. $(\sigma^{\underline{abc}})_{\alpha\beta}(\sigma_{\underline{abc}})_{\gamma\delta} = -2\cdot 3!\ (\sigma^{\underline{a}})_{\alpha[\gamma}(\sigma_{\underline{a}})_{\delta]\beta},$
7. $(\sigma^{\underline{abc}})^{\alpha\beta}(\sigma_{\underline{abc}})^{\gamma\delta} = -2\cdot 3!\ (\sigma^{\underline{a}})^{\alpha[\gamma}(\sigma_{\underline{a}})^{\delta]\beta},$
8. $(\sigma^{[4]})_{\alpha}{}^{\gamma}(\sigma_{[4]})_{\beta}{}^{\delta} = 4!\{-2\ \delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta} + 12\ \delta_{\beta}^{\gamma}\delta_{\alpha}^{\delta} - 2\ (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\},$
9. $(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta} = 5!\{-16\ \delta_{(\alpha}^{\gamma}\delta_{\beta)}^{\delta} - 2\ (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\},$
10. $(\sigma^{\underline{ab}})_{\alpha}{}^{\gamma}(\sigma_{\underline{ab}})_{\beta}{}^{\delta} = 2\ \{-\delta_{\alpha}^{\gamma}\delta_{\beta}^{\delta} - 4\ \delta_{\beta}^{\gamma}\delta_{\alpha}^{\delta} - 2\ (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\},$
11. $(\sigma^{\underline{ab}})_{[\alpha}{}^{\gamma}(\sigma_{\underline{ab}})_{\beta]}{}^{\delta} = 6\ \delta_{[\alpha}^{\gamma}\delta_{\beta]}^{\delta},$
12. $(\sigma^{\underline{ab}})_{(\alpha}{}^{\gamma}(\sigma_{\underline{ab}})_{\beta)}{}^{\delta} = -10\ \delta_{(\alpha}^{\gamma}\delta_{\beta)}^{\delta} - 8\ (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta},$
13. $[2\ \delta_{(\alpha}^{\gamma}\delta_{\beta}^{\delta} + (\sigma^{\underline{b}})_{(\alpha\beta|}(\sigma_{\underline{b}})^{\gamma\delta}](\sigma_{\underline{a}})_{\epsilon)\delta} = 0,$
14. $\{(\sigma_{[\underline{a}})_{(\alpha|\gamma}(\sigma^{\underline{efg}})^{\gamma\delta} + \frac{1}{2}(\sigma^{\underline{efg}})_{(\alpha|\gamma}(\sigma_{[\underline{a}})^{\gamma\delta}\}(\sigma_{bcd]})_{\beta)\delta} = 48\ (\sigma_{[a})_{\alpha\beta}\ \delta_{\underline{a}}^{\underline{e}}\delta_{\underline{b}}^{\underline{f}}\delta_{\underline{c}]}^{\underline{g}},$
15. $(\sigma_{\underline{abc}})_{(\alpha(\beta}(\sigma^{\underline{c}})_{\gamma)\delta)} = -2\ (\sigma_{[\underline{a}})_{\alpha\delta}(\sigma_{\underline{b}]})_{\beta\gamma},$
16. $(\sigma^{\underline{a}})_{(\alpha\beta}(\sigma_{\underline{ab}})_{\gamma)}{}^{\delta} = -(\sigma_{\underline{b}})_{(\alpha\beta}\delta_{\gamma)}^{\delta},$
17. $(\sigma^{\underline{a}})^{(\alpha\beta}(\sigma_{\underline{ab}})_{\delta}{}^{\gamma)} = (\sigma_{\underline{b}})^{(\alpha\beta}\delta_{\delta}^{\gamma)},$
18. $(\sigma^{\underline{c}})_{(\alpha\beta}(\sigma_{\underline{ab}})_{\gamma)}{}^{\delta}(\sigma_{\underline{c}})_{\delta\epsilon} = -2(\sigma_{[\underline{a}})_{(\alpha\beta}(\sigma_{\underline{b}]})_{\gamma)\epsilon},$
19. $(\sigma^{\underline{d}})_{(\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma)}{}^{\delta} = \frac{1}{2}(\sigma_{[\underline{a}})_{(\alpha\beta}(\sigma_{\underline{bc}]})_{\gamma)}{}^{\delta},$
20. $(\sigma_{[\underline{a}})_{(\alpha\beta}(\sigma_{\underline{bc}]})_{\gamma)}{}^{\delta} = -(\sigma_{\underline{d}})_{(\alpha\beta}(\sigma^{\underline{d}})^{\delta\epsilon}(\sigma_{\underline{abc}})_{\gamma)\epsilon},$

The following are some identities derivable from the definitions above.

19. $(\sigma_{\underline{bcd}})_{\alpha\beta}(\sigma^{\underline{abc}})^{\beta\gamma} \;=\; 72\;\delta_{\underline{d}}^{\underline{a}}\delta_\alpha^\gamma \;+\; 56(\sigma_{\underline{d}}{}^{\underline{a}})_\alpha{}^\gamma,$

20. $(\sigma^{\underline{abcd}})_\alpha{}^\beta(\sigma_{\underline{bcd}})_{\beta\gamma} \;=\; -\,7\cdot 8\cdot 9\;(\sigma^{\underline{a}})_{\alpha\gamma},$

21. $(\sigma^{\underline{abc}})^{\alpha\beta}(\sigma_{\underline{c}})_{\beta\gamma} \;=\; 8\;(\sigma^{\underline{ab}})_\gamma{}^\alpha,$

22. $(\sigma^{\underline{abc}})_{\alpha\beta}(\sigma_{\underline{c}})^{\beta\gamma} \;=\; -\,8\;(\sigma^{\underline{ab}})_\alpha{}^\gamma,$

23. $(\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{\underline{ab}})_\gamma{}^\beta \;=\; -\,9\;(\sigma^{b})_{\alpha\gamma},$

24. $(\sigma_{\underline{a}})^{\alpha\beta}(\sigma^{\underline{ab}})_\beta{}^\gamma \;=\; 9\;(\sigma^{b})^{\alpha\gamma},$

25. $(\sigma^{\underline{abcd}})_\alpha{}^\beta(\sigma_{\underline{a}})_{\beta\gamma} \;=\; -\,7\;(\sigma^{\underline{bcd}})_{\alpha\gamma},$

26. $(\sigma^{\underline{abcd}})_\beta{}^\alpha(\sigma_{\underline{a}})^{\beta\gamma} \;=\; -\,7\;(\sigma^{\underline{bcd}})^{\alpha\gamma},$

27. $(\sigma_{\underline{ab}})_\alpha{}^\beta(\sigma^{\underline{cd}})_\beta{}^\gamma \;=\; -\;\delta_{[\underline{a}}^{\underline{c}}\delta_{\underline{b}]}^{\underline{d}}\delta_\alpha^\gamma - \delta_{[\underline{a}}^{[\underline{c}}(\sigma_{\underline{b}]}{}^{\underline{d}]})_\alpha{}^\gamma + (\sigma_{\underline{ab}}{}^{\underline{cd}})_\alpha{}^\gamma,$

28. $(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{def}})^{\beta\gamma} \;=\; \delta_{[\underline{a}}^{\underline{d}}\delta_{\underline{b}}^{\underline{e}}\delta_{\underline{c}]}^{\underline{f}}\delta_\alpha^\gamma + \frac{1}{2}\delta_{[\underline{a}}^{[\underline{d}}\delta_{\underline{b}}^{\underline{e}}(\sigma_{\underline{c}]}{}^{\underline{f}]})_\alpha{}^\gamma - \frac{1}{4}\delta_{[\underline{a}}^{[\underline{d}}(\sigma_{\underline{bc}]}{}^{\underline{ef}]})_\alpha{}^\gamma - (\sigma_{\underline{abc}}{}^{\underline{def}})_\alpha{}^\gamma,$

29. $(\sigma^{\underline{ab}})_\alpha{}^\beta(\sigma_{\underline{abc}})_{\beta\gamma} \;=\; -\,72\;(\sigma_{\underline{c}})_{\alpha\gamma},$

30. $(\sigma_{\underline{abc}})^{\alpha\beta}(\sigma^{\underline{ab}})_\beta{}^\gamma \;=\; -\,72\;(\sigma_{\underline{c}})^{\alpha\gamma},$

31. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{ab}})_\beta{}^\gamma \;=\; -\,6\cdot 7\;(\sigma^{\underline{cde}})^{\alpha\gamma},$

32. $(\sigma^{\underline{abcde}})_{\alpha\beta}(\sigma_{\underline{ab}})_\gamma{}^\beta \;=\; 6\cdot 7\;(\sigma^{\underline{cde}})_{\alpha\gamma},$

33. $(\sigma^{\underline{ab}})_\alpha{}^\beta(\sigma_{\underline{ab}}{}^{\underline{cd}})_\beta{}^\gamma \;=\; -\,7\cdot 8\;(\sigma^{\underline{cd}})_\alpha{}^\gamma,$

34. $(\sigma^{\underline{ab}})_\beta{}^\alpha(\sigma_{\underline{ab}}{}^{\underline{cd}})_\gamma{}^\beta \;=\; -\,7\cdot 8\;(\sigma^{\underline{cd}})_\gamma{}^\alpha,$

35. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{abc}})_{\beta\gamma} \;=\; -\,6\cdot 7\cdot 8\;(\sigma^{\underline{de}})_\gamma{}^\alpha,$

36. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{b\underline{cde}})_\beta{}^\gamma \;=\; 6\cdot 7\cdot 8\cdot 9\;(\sigma^{\underline{a}})^{\alpha\gamma},$

37. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{a}})_{\beta\gamma} \;=\; -\,6\;(\sigma^{\underline{bcde}})_\gamma{}^\alpha,$

38. $(\sigma^{\underline{abcde}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma} \;=\; -\,6\;(\sigma^{\underline{bcde}})_\alpha{}^\gamma,$

39. $(\sigma^{\underline{ab}}{}_{\underline{c}})^{\alpha\beta}(\sigma_{\underline{abd}})_{\alpha\gamma} \;=\; 7\cdot 8\;(\sigma_{\underline{cd}})_\gamma{}^\beta - 8\cdot 9\eta_{\underline{cd}}\delta_\gamma^\beta,$

40. $(\sigma^{\underline{f}}{}_{\underline{ab}})^{\alpha\beta}(\sigma_{\underline{fcd}})_{\alpha\gamma} \;=\; 6\;(\sigma_{\underline{abcd}})_\gamma{}^\beta - 7\eta_{\underline{a}[\underline{c}}(\sigma_{\underline{d}]\underline{b}})_\gamma{}^\beta + 7\eta_{\underline{b}[\underline{c}}(\sigma_{\underline{d}]\underline{a}})_\gamma{}^\beta + 8\eta_{\underline{a}[\underline{c}}\eta_{\underline{d}]\underline{b}}\delta_\gamma^\beta,$

41. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{abf}})_{\beta\gamma} \;=\; -\,5\cdot 6\;(\sigma_{\underline{f}}{}^{\underline{cde}})_\gamma{}^\alpha - \frac{6\cdot 7}{2}\delta_{\underline{f}}^{[c}(\sigma^{de]})_\gamma{}^\alpha,$

42. $(\sigma^{\underline{abc}})^{\alpha\beta}(\sigma_{[4]})_\beta{}^\gamma(\sigma_{\underline{ab}})_\gamma{}^\delta \;=\; 4\;(\sigma^{\underline{c}})^{\alpha\beta}(\sigma_{[4]})_\beta{}^\delta \;+\; 4\;(\sigma_{[4]})_\beta{}^\alpha(\sigma^{\underline{c}})^{\beta\delta},$

43. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{afg}})_{\beta\gamma} \;=\; -\,4\;(\sigma_{\underline{fg}}{}^{\underline{bcde}})_\gamma{}^\alpha \qquad if \quad \underline{bcde} \neq \underline{gf},$

44. $(\sigma_{\underline{c}}{}^{\underline{ab}})^{\alpha\beta}(\sigma_{\underline{ad}})_\beta{}^\gamma(\sigma_{\underline{eb}})_\alpha{}^\delta \;=\; 3\cdot 19\ (\sigma_{\underline{cde}})^{\gamma\delta}$ $\quad if \quad \underline{c}\neq\underline{d}\neq\underline{e},$

45. $(\sigma^{\underline{e}}{}_{\underline{ab}})^{\alpha\beta}(\sigma_{\underline{ecd}})_{\alpha\gamma} \;=\; 6\ (\sigma_{\underline{abcd}})_\gamma{}^\beta$ $\quad if \quad [\underline{ab}]\neq[\underline{cd}],$

46. $(\sigma^{\underline{abcde}})^{\alpha\beta}(\sigma_{\underline{af}})_\beta{}^\gamma(\sigma_{\underline{bg}})_\alpha{}^\delta \;=\; 31(\sigma_{\underline{fg}}{}^{\underline{cde}})^{\gamma\delta}$ $\quad if \quad \underline{f}\neq\underline{g}\neq[\underline{cde}]\neq\underline{f},$

47. $(\sigma^{\underline{ab}}{}_{\underline{cde}})^{\alpha\beta}(\sigma_{\underline{a}\underline{fg}})_{\beta\gamma} \;=\; 4\ (\sigma^{\underline{b}}{}_{\underline{cde}\underline{fg}})_\gamma{}^\alpha - 5\ \delta^{\underline{b}}_{[\underline{f}}(\sigma_{\underline{g}]\underline{cde}})_\gamma{}^\alpha$ $\quad if \quad \underline{f}\neq\underline{g}\neq[\underline{cde}],$

48. $(\sigma^{\underline{abc}})_{\alpha\gamma}(\sigma_{\underline{de}})_\beta{}^\gamma \;=\; -\ (\sigma^{\underline{abc}}{}_{\underline{de}})_{\alpha\beta} - \frac{1}{2}\delta^{[\underline{a}}_{[\underline{d}}(\sigma^{\underline{bc}]}{}_{\underline{e}]})_{\alpha\beta} + \delta^{[\underline{a}}_{\underline{d}}\delta^{\underline{b}}_{\underline{e}}(\sigma^{\underline{c}]})_{\alpha\beta},$

49. $(\sigma^{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{de}})_\gamma{}^\beta \;=\; (\sigma^{\underline{abc}}{}_{\underline{de}})^{\alpha\beta} + \frac{1}{2}\delta^{[\underline{a}}_{[\underline{d}}(\sigma^{\underline{bc}]}{}_{\underline{e}]})^{\alpha\beta} - \delta^{[\underline{a}}_{\underline{d}}\delta^{\underline{b}}_{\underline{e}}(\sigma^{\underline{c}]})^{\alpha\beta},$

50. $(\sigma_{\underline{abcde}\underline{f}})_\beta{}^\alpha(\sigma_{\underline{g}})^{\beta\gamma} \;=\; (\sigma_{\underline{abcde}\underline{fg}})^{\alpha\gamma} - \frac{1}{5!}\eta_{g[\underline{a}}(\sigma_{\underline{bcde}\underline{f}]})^{\alpha\gamma},$

51. $(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{bcd}})^{\beta\gamma}(\sigma^{\underline{e}})_{\gamma\delta} \;=\; -\,\eta^{\underline{ae}}(\sigma_{\underline{bcd}})_{\alpha\delta} - \frac{1}{2}\delta^{[\underline{a}}_{[\underline{b}}(\sigma_{\underline{cd}]}{}^{\underline{e}]})_{\alpha\delta} + (\sigma^{\underline{ae}}{}_{\underline{bcd}})_{\alpha\delta} + \delta^{\underline{a}}_{[\underline{b}}\delta^{\underline{e}}_{\underline{c}}(\sigma_{\underline{d}]})_{\alpha\delta},$

52. $(\sigma_{[3]})^{\alpha\beta}(\sigma_{\underline{ab}})_\beta{}^\gamma(\sigma_{\underline{cd}})_\alpha{}^\delta \;=\; \frac{1}{3!}\epsilon_{\underline{abcd}[3][\bar{3}]}(\sigma^{[\bar{3}]})^{\gamma\delta}$ $\quad if \quad [3]\neq[\underline{a},\underline{b},\underline{c},\underline{d}],$

55. $(\sigma^{\underline{ab}})_\alpha{}^\beta(\sigma_{\underline{c}})_{\beta\gamma}(\sigma_{\underline{ab}})_\delta{}^\gamma \;=\; 54\ (\sigma_{\underline{c}})_{\alpha\delta},$

56. $(\sigma^{\underline{ab}})_\alpha{}^\beta(\sigma^{\underline{cd}})_\beta{}^\gamma(\sigma_{\underline{ab}})_\gamma{}^\delta \;=\; -\,26\ (\sigma^{\underline{cd}})_\alpha{}^\delta,$

57. $(\sigma^{\underline{abc}})_{\alpha\gamma}(\sigma_{\underline{dc}})_\beta{}^\gamma \;=\; 7\ (\sigma_{\underline{d}}{}^{\underline{ab}})_{\alpha\beta} - 8\ \delta^{[\underline{a}}_{\underline{d}}(\sigma^{\underline{b}]})_{\alpha\beta},$

58. $(\sigma_{\underline{ac}})_\beta{}^\alpha(\sigma^{\underline{ab}})_\gamma{}^\beta \;=\; -\,8\ (\sigma^{\underline{b}}{}_{\underline{c}})_\gamma{}^\alpha - 9\ \delta^{\underline{b}}_{\underline{c}}\delta^\alpha_\gamma,$

59. $(\sigma_{\underline{ab}})_\gamma{}^\beta(\sigma^{[4]})_\beta{}^\delta(\sigma^{\underline{b}})_{\delta\alpha} \;=\; 2\ (\sigma_{\underline{a}})_{\gamma\beta}(\sigma^{[4]})_\alpha{}^\beta - (\sigma^{[4]})_\gamma{}^\beta(\sigma_{\underline{a}})_{\beta\alpha},$

60. $(\sigma^{\underline{bc}})_\gamma{}^\beta(\sigma^{[4]})_\beta{}^\delta(\sigma_{\underline{abc}})_{\delta\alpha} \;=\; 2\ (\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{[4]})_\gamma{}^\beta + 4\ (\sigma_{\underline{a}})_{\gamma\beta}(\sigma^{[4]})_\alpha{}^\beta,$

61. $(\sigma^{\underline{abcd}})_\gamma{}^\beta(\sigma^{[4]})_\beta{}^\delta(\sigma_{\underline{bcd}})_{\delta\alpha} \;=\; 36\ (\sigma^{\underline{a}})_{\gamma\beta}(\sigma^{[4]})_\alpha{}^\beta - 6\ (\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{[4]})_\gamma{}^\beta,$

62. $\delta^\gamma_{[\alpha}\delta^\delta_{\beta]} \;=\; -\,\frac{1}{48}(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{abc}})^{\gamma\delta},$

63. $(\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} \;=\; -\,10\ \delta^\gamma_\alpha,$

64. $(\sigma_{\underline{ab}})_\alpha{}^\beta(\sigma^{\underline{ab}})_\beta{}^\gamma \;=\; -\,10\cdot 9\ \delta^\gamma_\alpha,$

65. $(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{abc}})^{\beta\gamma} \;=\; 10\cdot 9\cdot 8\ \delta^\gamma_\alpha,$

66. $(\sigma_{\underline{abcd}})_\alpha{}^\beta(\sigma^{\underline{abcd}})_\beta{}^\gamma \;=\; 10\cdot 9\cdot 8\cdot 7\ \delta^\gamma_\alpha,$

67. $(\sigma_{\underline{abcde}})_{\alpha\beta}(\sigma^{\underline{abcde}})^{\beta\gamma} \;=\; 10\cdot 9\cdot 8\cdot 7\cdot 6\ \delta^\gamma_\alpha,$

68. $(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} \;=\; 8\ (\sigma_{\underline{a}})_{\alpha\delta},$

69. $(\sigma^{\underline{f}})^{\alpha\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}})^{\gamma\delta} \;=\; 8\;(\sigma_{\underline{a}})^{\alpha\delta},$

70. $(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{ab}})_{\gamma}{}^{\beta}(\sigma_{\underline{f}})^{\gamma\delta} \;=\; 6\;(\sigma_{\underline{ab}})_{\alpha}{}^{\delta},$

71. $(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{abc}})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} \;=\; 4\;(\sigma_{\underline{abc}})_{\alpha\delta},$

72. $(\sigma^{\underline{f}})^{\alpha\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}})^{\gamma\delta} \;=\; 4\;(\sigma_{\underline{abc}})^{\alpha\delta},$

73. $(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma}{}^{\beta}(\sigma_{\underline{f}})^{\gamma\delta} \;=\; -\,2\;(\sigma_{\underline{abcd}})_{\alpha}{}^{\delta},$

74. $(\sigma^{\underline{f}})_{\alpha\beta}(\sigma_{[5]})^{\beta\gamma}(\sigma_{\underline{f}})_{\gamma\delta} \;=\; 0,$

75. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; 54\;(\sigma_{\underline{a}})_{\alpha\delta},$

76. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{a}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\gamma}{}^{\delta} \;=\; 54\;(\sigma_{\underline{a}})^{\beta\delta},$

77. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{ab}})_{\gamma}{}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; -\,26\;(\sigma_{\underline{ab}})_{\delta}{}^{\beta},$

78. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; 6\;(\sigma_{\underline{abc}})_{\alpha\delta},$

79. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\gamma}{}^{\delta} \;=\; 6\;(\sigma_{\underline{abc}})^{\beta\delta},$

80. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{abcd}})_{\gamma}{}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; 6\;(\sigma_{\underline{abcd}})_{\delta}{}^{\beta},$

81. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{abcde}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; -\,10\;(\sigma_{\underline{abcde}})_{\alpha\delta},$

82. $(\sigma^{\underline{f}_1\underline{f}_2})_{\alpha}{}^{\beta}(\sigma_{\underline{abcde}})^{\alpha\delta}(\sigma_{\underline{f}_1\underline{f}_2})_{\delta}{}^{\gamma} \;=\; -\,10\;(\sigma_{\underline{abcde}})^{\beta\gamma},$

83. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{a}})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} \;=\; -\,8\cdot 36\;(\sigma_{\underline{a}})_{\alpha\delta},$

84. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})^{\alpha\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})^{\gamma\delta} \;=\; -\,8\cdot 36\;(\sigma_{\underline{a}})^{\alpha\delta},$

85. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{ab}})_{\gamma}{}^{\beta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})^{\gamma\delta} \;=\; -\,48\;(\sigma_{\underline{ab}})_{\alpha}{}^{\delta},$

86. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{abc}})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} \;=\; 48\;(\sigma_{\underline{abc}})_{\alpha\delta},$

87. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})^{\alpha\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})^{\gamma\delta} \;=\; 48\;(\sigma_{\underline{abc}})^{\alpha\delta},$

88. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{\underline{abcd}})_{\gamma}{}^{\beta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})^{\gamma\delta} \;=\; -\,48\;(\sigma_{\underline{abcd}})_{\alpha}{}^{\delta},$

89. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3})_{\alpha\beta}(\sigma_{[5]})^{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3})_{\gamma\delta} \;=\; 0,$

90. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}{}^{\beta}(\sigma_{\underline{a}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}{}^{\gamma} \;=\; 24\cdot 42\;(\sigma_{\underline{a}})_{\alpha\delta},$

91. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}{}^{\beta}(\sigma_{\underline{a}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}{}^{\gamma} \;=\; 24\cdot 42\;(\sigma_{\underline{a}})^{\beta\delta},$

92. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}{}^{\beta}(\sigma_{\underline{ab}})_{\gamma}{}^{\alpha}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}{}^{\gamma} \;=\; -\,24\cdot 14\;(\sigma_{\underline{ab}})_{\delta}{}^{\beta},$

93. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\alpha}{}^{\beta}(\sigma_{\underline{abc}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_{\delta}{}^{\gamma} \;=\; -\,24\cdot 14\;(\sigma_{\underline{abc}})_{\alpha\delta},$

94. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\alpha{}^\beta(\sigma_{\underline{abc}})^{\alpha\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\gamma{}^\delta \;=\; -\,24\cdot 14\;(\sigma_{\underline{abc}})^{\beta\delta},$

95. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\alpha{}^\beta(\sigma_{\underline{abcd}})_\gamma{}^\alpha(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\delta{}^\gamma \;=\; 48\;(\sigma_{\underline{abcd}})_\delta{}^\beta,$

96. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\alpha{}^\beta(\sigma_{\underline{abcde}})_{\beta\gamma}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\delta{}^\gamma \;=\; 240\;(\sigma_{\underline{abcde}})_{\alpha\delta},$

97. $(\sigma^{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\alpha{}^\beta(\sigma_{\underline{abcde}})^{\alpha\delta}(\sigma_{\underline{f}_1\underline{f}_2\underline{f}_3\underline{f}_4})_\delta{}^\gamma \;=\; 240\;(\sigma_{\underline{abcde}})^{\beta\gamma}.$

11D Conventions and Notation

The conventions that we use for 11D superspace can be chosen so that we use real (i.e. Majorana) 32-component spinors for the Grassmann coordinates of superspace. Our γ-matrices are defined by

$$\{ \gamma^a \, , \gamma^b \} \; = \; 2 \, \eta^{ab} \, \mathrm{I} \quad , \tag{B.1}$$

where the signature of the metric is the "mostly minus one," i. e. diag. $(+, -, \cdots, -)$. This implies that our gamma matrices satisfy the complex conjugation conditions

$$\begin{aligned} [\, (\gamma^a)_\alpha{}^\beta \,]^* &= \; - \; (\gamma^a)_\alpha{}^\beta \quad , \quad [\, (\gamma^{[2]})_\alpha{}^\beta \,]^* \; = \; (\gamma^{[2]})_\alpha{}^\beta \quad , \\ [\, (\gamma^{[3]})_\alpha{}^\beta \,]^* &= \; - \; (\gamma^{[3]})_\alpha{}^\beta \quad , \quad [\, (\gamma^{[4]})_\alpha{}^\beta \,]^* \; = \; (\gamma^{[4]})_\alpha{}^\beta \quad , \\ [\, (\gamma^{[5]})_\alpha{}^\beta \,]^* &= \; - \; (\gamma^{[5]})_\alpha{}^\beta \quad . \end{aligned} \tag{B.2}$$

Our 'spinor metric,' with which we raise and lower spinor indices, is denoted by $C_{\alpha\beta}$ and satisfies,

$$C_{\alpha\beta} \; = \; - \, C_{\beta\alpha} \quad , \quad [\, C_{\alpha\beta} \,]^* \; = \; - \, C_{\alpha\beta} \quad . \tag{B.3}$$

The inverse spinor metric $C^{\alpha\beta}$ is defined to satisfy

$$C_{\alpha\beta} \, C^{\gamma\beta} \; = \; \delta_\alpha{}^\gamma \quad . \tag{B.4}$$

We also use superspace conjugation which permits the appearance of appropriate factors of i even within a theory of involving solely real spinors. A complete discussion of superspace conjugation can be found in my TASI lectures.

We define our gamma matrices with multiple numbers of vector indices through the equations

$$\begin{aligned} \gamma_a \, \gamma_b &= \; \gamma_{a\,b} \; + \; \eta_{a\,b} \quad , \qquad \gamma_b \, \gamma_a \; = \; - \, \gamma_{a\,b} \; + \; \eta_{a\,b} \quad , \\ \gamma_a \, \gamma_{b\,c} &= \; \gamma_{a\,b\,c} \; + \; \eta_{a\,[b} \gamma_{c]} \quad , \qquad \gamma_{b\,c} \, \gamma_a \; = \; \gamma_{a\,b\,c} \; - \; \eta_{a\,[b} \gamma_{c]} \quad , \\ \gamma_a \, \gamma_{b\,c\,d} &= \; \gamma_{a\,b\,c\,d} \; + \; \tfrac{1}{2} \eta_{a\,[b|} \gamma_{|c\,d]} \quad , \qquad \gamma_{b\,c\,d} \, \gamma_a \; = \; - \, \gamma_{a\,b\,c\,d} \; + \; \tfrac{1}{2} \eta_{a\,[b|} \gamma_{|c\,d]} \quad , \\ \gamma_a \, \gamma_{b\,c\,d\,e} &= \; \gamma_{a\,b\,c\,d\,e} \; + \; \tfrac{1}{6} \eta_{a\,[b|} \gamma_{|c\,d\,e]} \quad , \qquad \gamma_{b\,c\,d\,e} \, \gamma_a \; = \; \gamma_{a\,b\,c\,d\,e} \; - \; \tfrac{1}{6} \eta_{a\,[b|} \gamma_{|c\,d\,e]} \quad , \\ \gamma_a \, \gamma_{b\,c\,d\,e\,f} &= \; i \, \frac{1}{120} \, \epsilon_{a\,b\,c\,d\,e\,f}{}^{[5]} \gamma_{[5]} \; + \; \frac{1}{24} \eta_{a\,[b|} \gamma_{|c\,d\,e\,f]} \quad , \\ \gamma_{b\,c\,d\,e\,f} \, \gamma_a &= \; - \, i \, \frac{1}{120} \, \epsilon_{a\,b\,c\,d\,e\,f}{}^{[5]} \gamma_{[5]} \; + \; \frac{1}{24} \eta_{a\,[b|} \gamma_{|c\,d\,e\,f]} \quad . \end{aligned} \tag{B.5}$$

The basic identities for non-vanishing traces over the gamma matrices are

$$
\begin{aligned}
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_a\,\gamma^b\,) \;&=\; \delta_a{}^b \quad , \\
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_{a_1a_2}\,\gamma^{b_1b_2}\,) \;&=\; -\,\delta_{[a_1|}{}^{b_1}\,\delta_{|a_2]}{}^{b_2} \quad , \\
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_{a_1a_2a_3}\,\gamma^{b_1b_2b_3}\,) \;&=\; -\,\delta_{[a_1|}{}^{b_1}\,\delta_{|a_2|}{}^{b_2}\,\delta_{|a_3]}{}^{b_3} \quad , \\
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_{a_1a_2a_3a_4}\,\gamma^{b_1b_2b_3b_4}\,) \;&=\; \delta_{[a_1|}{}^{b_1}\,\delta_{|a_2|}{}^{b_2}\,\delta_{|a_3|}{}^{b_3}\,\delta_{|a_4]}{}^{b_4} \quad , \\
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_{a_1a_2a_3a_4a_5}\,\gamma^{b_1b_2b_3b_4b_5}\,) \;&=\; \delta_{[a_1|}{}^{b_1}\,\delta_{|a_2|}{}^{b_2}\,\delta_{|a_3|}{}^{b_3}\,\delta_{|a_4|}{}^{b_4}\,\delta_{|a_5]}{}^{b_5} \quad , \\
\frac{1}{32}\,\mathrm{Tr}\,(\,\gamma_{a_1}\,\cdots\,\gamma_{a_{11}})\;&=\; i\,\epsilon_{a_1\ldots a_{11}} \quad .
\end{aligned}
\tag{B.6}
$$

Using the spinor metric to lower one spinor index of the quantities in (B.2) we find

$$
\begin{aligned}
[\,(\gamma^a)_{\alpha\beta}\,]^* &= (\gamma^a)_{\alpha\beta} \quad , \quad [\,(\gamma^{[2]})_{\alpha\beta}\,]^* \;=\; -\,(\gamma^{[2]})_{\alpha\beta} \quad , \\
[\,(\gamma^{[3]})_{\alpha\beta}\,]^* &= (\gamma^{[3]})_{\alpha\beta} \quad , \quad [\,(\gamma^{[4]})_{\alpha\beta}\,]^* \;=\; -\,(\gamma^{[4]})_{\alpha\beta} \quad , \\
[\,(\gamma^{[5]})_{\alpha\beta}\,]^* &= (\gamma^{[5]})_{\beta\alpha} \quad ,
\end{aligned}
\tag{B.7}
$$

and as well the same equations apply to the matrices with two raised spinor indices. In addition these satisfy the symmetry relations

$$
\begin{aligned}
&(\gamma^a)_{\alpha\beta} = (\gamma^a)_{\beta\alpha} \quad , \quad (\gamma^{[2]})_{\alpha\beta} \;=\; (\gamma^{[2]})_{\beta\alpha} \quad , \quad (\gamma^{[5]})_{\alpha\beta} \;=\; (\gamma^{[5]})_{\beta\alpha} \quad , \\
&(\gamma^{[3]})_{\alpha\beta} = \;-\,(\gamma^{[3]})_{\beta\alpha} \quad , \quad (\gamma^{[4]})_{\alpha\beta} \;=\; -\,(\gamma^{[4]})_{\beta\alpha} \quad ,
\end{aligned}
\tag{B.8}
$$

where the same equations apply to the matrices with two raised spinor indices.

Other identities on the 11D gamma matrices include

$$
\gamma^{[P]}\,\gamma_{[Q]}\,\gamma_{[P]} \;=\; c_{[Q]\,[P]}\,\gamma_{[Q]} \quad , \tag{B.9}
$$

where the coefficients $c_{[Q]\,[P]}$ are given in the following table.

	$[P]=[1]$	$[P]=[2]$	$[P]=[3]$	$[P]=[4]$	$[P]=[5]$
$[Q]=[1]$	-9	-70	450	$2,160$	$-5,040$
$[Q]=[2]$	7	-38	-126	-144	$-5,040$
$[Q]=[3]$	-5	-14	-30	-528	$1,680$
$[Q]=[4]$	3	2	66	-144	$1,680$
$[Q]=[5]$	-1	10	-30	240	$-1,200$

For example this table implies

$$
\gamma^{[3]}\,\gamma_{[2]}\,\gamma_{[3]} \;=\; -\,126\,\gamma_{[2]} \quad , \tag{B.10}
$$

Some useful Fierz-type identities include the following

$$
\begin{aligned}
0 &= (\gamma^a)_{(\alpha\beta|}(\gamma_{ab})_{|\gamma\delta)} \quad , \\
0 &= 5(\gamma^a)_{(\alpha\beta|}(\gamma_a)_{|\gamma\delta)} \;+\; \tfrac{1}{2}(\gamma^{[2]})_{(\alpha\beta|}(\gamma_{[2]})_{|\gamma\delta)} \quad , \\
0 &= 6(\gamma^a)_{(\alpha\beta|}(\gamma_a)_{|\gamma\delta)} \;+\; \tfrac{1}{5!}(\gamma^{[5]})_{(\alpha\beta|}(\gamma_{[5]})_{|\gamma\delta)} \quad , \\
0 &= (\gamma^e)_{(\alpha\beta|}(\gamma_{abcde})_{|\gamma\delta)} \;-\; \tfrac{1}{8}(\gamma_{[ab|})_{(\alpha\beta|}(\gamma_{|cd]})_{|\gamma\delta)} \quad , \\
0 &= \tfrac{1}{2}(\gamma^{ab})_{(\alpha\beta|}(\gamma_{abcde})_{|\gamma\delta)} \;+\; 2(\gamma_{[c|})_{(\alpha\beta|}(\gamma_{|de]})_{|\gamma\delta)} \quad , \\
0 &= (\gamma^a)_{(\alpha\beta|}(\gamma_a)_{|\gamma)}{}^{\delta} \;-\; \tfrac{1}{2}(\gamma^{[2]})_{(\alpha\beta|}(\gamma_{[2]})_{|\gamma)}{}^{\delta} \;+\; \tfrac{1}{5!}(\gamma^{[5]})_{(\alpha\beta|}(\gamma_{[5]})_{|\gamma)}{}^{\delta} \quad .
\end{aligned}
\tag{B.11}
$$

Our Lorentz generator is defined to realize

$$
\begin{aligned}
&[\, \mathcal{M}_{ab} \,,\, (\gamma_a)_{\alpha\beta} \,\} \;=\; 0 \quad , \\
&[\, \mathcal{M}_{ab} \,,\, \nabla_\alpha \,\} \;=\; \tfrac{1}{2}\,(\gamma_{ab})_\alpha{}^\beta\, \nabla_\beta \quad , \quad [\, \mathcal{M}_{ab} \,,\, \nabla_c \,\} \;=\; \eta_{ca}\, \nabla_b \;-\; \eta_{cb}\, \nabla_a \quad , \\
&[\, \mathcal{M}_{ab} \,,\, \mathcal{M}_{cd} \,\} \;=\; \eta_{ca}\, \mathcal{M}_{bd} \;-\; \eta_{cb}\, \mathcal{M}_{ad} \;-\; \eta_{da}\, \mathcal{M}_{bc} \;+\; \eta_{db}\, \mathcal{M}_{ac} \quad .
\end{aligned}
\tag{B.12}
$$

One notational device not discussed in our previous paper is the definition of the 11D super-epsilon tensor, which we denote by $\widehat{\epsilon}_{A_1 \dots A_4}{}^{B_1 \dots B_7}$, and define by

$$
\begin{aligned}
\widehat{\epsilon}_{a_1 \dots a_4}{}^{b_1 \dots b_7} &\equiv \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \quad , \\
\widehat{\epsilon}_{\alpha_1\, a_2 \dots a_4}{}^{\beta_1\, b_2 \dots b_7} &\equiv (\gamma^{a_1}{}_{b_1})_{\alpha_1}{}^{\beta_1}\, \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \quad , \\
\widehat{\epsilon}_{\alpha_1\, \alpha_2\, a_3\, a_4}{}^{\beta_1\, \beta_2\, b_3 \dots b_7} &\equiv (\gamma^{a_1}{}_{b_1})_{(\alpha_1|}{}^{\beta_1}\, (\gamma^{a_2}{}_{b_2})_{|\alpha_2)}{}^{\beta_2}\, \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \quad , \\
\widehat{\epsilon}_{\alpha_1\, \alpha_2\, \alpha_3\, a_4}{}^{\beta_1\, \beta_2\, \beta_3\, b_4\, b_5\, b_6\, b_7} &\equiv (\gamma^{a_1}{}_{b_1})_{(\alpha_1|}{}^{\beta_1}\, (\gamma^{a_2}{}_{b_2})_{|\alpha_2|}{}^{\beta_2}\, (\gamma^{a_3}{}_{b_3})_{|\alpha_3)}{}^{\beta_3}\, \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \quad , \\
\widehat{\epsilon}_{\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4}{}^{\beta_1\, \beta_2\, \beta_3\, \beta_4\, b_5\, b_6\, b_7} &\equiv (\gamma^{a_1}{}_{b_1})_{(\alpha_1|}{}^{\beta_1}\, (\gamma^{a_2}{}_{b_2})_{|\alpha_2|}{}^{\beta_2}\, (\gamma^{a_3}{}_{b_3})_{|\alpha_3|}{}^{\beta_3}\, (\gamma^{a_4}{}_{b_4})_{|\alpha_4)}{}^{\beta_4}\, \epsilon_{a_1 \dots a_4}{}^{b_1 \dots b_7} \quad .
\end{aligned}
\tag{B.13}
$$

The role of this object is that it allows us to convert the components of a super 7-form $X_{A_1 \dots A_7}$ into the dual components of a super 4-form $\widehat{X}_{A_1 \dots A_4}$ via the definitions

$$
\begin{aligned}
\widehat{X}_{a_1 \dots a_4} &\equiv \tfrac{1}{7!}\widehat{\epsilon}_{a_1 \dots a_4}{}^{b_1 \dots b_7} X_{b_1 \dots b_7} \quad , \\
\widehat{X}_{\alpha_1 a_2 a_3 a_4} &\equiv \tfrac{1}{6!}\widehat{\epsilon}_{\alpha_1 a_2 a_3 a_4}{}^{\beta_1 b_2 \dots b_7} X_{\beta_1 b_2 \dots b_7} \quad , \\
\widehat{X}_{\alpha_1 \alpha_2 a_3 a_4} &\equiv \tfrac{1}{2\cdot 5!}\widehat{\epsilon}_{\alpha_1 \alpha_2 a_3 a_4}{}^{\beta_1 \beta_2 b_3 \dots b_7} X_{\beta_1 \beta_2 b_3 \dots b_7} \quad , \\
\widehat{X}_{\alpha_1 \alpha_2 \alpha_3 a_4} &\equiv \tfrac{1}{3!\cdot 4!}\widehat{\epsilon}_{\alpha_1 \alpha_2 \alpha_3 a_4}{}^{\beta_1 \beta_2 \beta_3 b_4 \dots b_7} X_{\beta_1 \beta_2 \beta_3 b_4 \dots b_7} \quad , \\
\widehat{X}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} &\equiv \tfrac{1}{4!\cdot 3!}\widehat{\epsilon}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}{}^{\beta_1 \beta_2 \beta_3 \beta_4 b_5 b_7 b_7} X_{\beta_1 \beta_2 \beta_3 \beta_4 b_5 b_6 b_7} \quad .
\end{aligned}
\tag{B.14}
$$

As well, it can be used for the reverse purpose of converting the components of a super 4-form to those of a super 7-form. The super-epsilon tensor concept has proven very useful for 10D theories as we suspect will also be the case for 11D theories. Similar construction follow for the case of 10D theories.