# Noncommutativity in String Theory

**Sunil Mukhi**
*IAS, Princeton and Tata Institute, Mumbai*

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1. Introduction and Motivation
2. Open String Effective Actions
3. Noncommutative Open String Actions
4. Properties of Noncommutative Gauge Theory
5. Open Wilson Lines
6. Gauge Invariant Noncommutative Actions
7. Summary of Further Directions

*Selected References:*

*A. Abouelsaood, C. Callan, C. Nappi, J. Yost,* "Open Strings in Background Gauge Fields" Nucl.Phys. B280 (599) 1987.

*N. Seiberg and E. Witten,* "String Theory and Noncommutative Geometry" hep-th/9908142.

*S.R. Das and S.-J. Rey,* "Open Wilson Lines in Noncommutative Gauge Theory and Tomography of Holographic Dual Supergravity" hep-th/0008042.

*D.J. Gross, A. Hashimoto and N. Itzhaki,* "Observables of Non-Commutative Gauge Theories" hep-th/0008075.

*H. Liu,* "*-Trek II: $*_n$ Operations, Open Wilson Lines and the Seiberg-Witten Map" hep-th/0011125.

*S.R. Das and S. Trivedi,* "Supergravity Couplings to Noncommutative Branes, Open Wilson Lines and Generalized Star Products" hep-th/0011131.

*S. Mukhi and N.V. Suryanarayana,* "Gauge-Invariant Couplings of Noncommutative Branes to Ramond-Ramond Backgrounds" hep-th/0104045.

*S.R. Das, S. Mukhi and N.V. Suryanarayana,* "Derivative Corrections from Noncommutativity" hep-th/0106024.

## 1. Introduction and Motivation

- Much of the dynamics of string theory is encoded in the low energy effective action.

  This action describes the coupling of light modes of the string.

- Because we are interested in theories of both open and closed strings, we can consider open string as well as closed string effective actions.

  But there are also couplings between open and closed string modes.

  In these lectures we will study the bosonic part of the open string effective action including the coupling to closed string modes. We will not study the purely closed string action.

- We will work in type IIB superstring theory, with a Euclidean spacetime. This is mainly for notational convenience.

- The bosonic light closed-string modes in type IIB string theory are:

$$\text{NS-NS:} \qquad g_{ij}, B_{ij}, \varphi$$
$$\text{R-R:} \qquad C^{(0)}, C^{(2)}_{i_1 i_2}, C^{(4)}_{i_1 \ldots i_4}, C^{(6)}_{i_1 \ldots i_6}, C^{(8)}_{i_1 \ldots i_8}, C^{(10)}_{i_1 \ldots i_{10}}$$

- To have open-string modes, we need a D-brane. Let us take it to be a Euclidean D9-brane, spanning all 10 dimensions. Then the bosonic light open-string modes are:

$$A_i$$

- We will study the coupling of $A_i$ to the NS-NS fields listed above. These will include the self-couplings of $A_i$. Hence the object of our study will be the action:

$$S_{\text{NS-NS}}[A_i; g_{ij}, B_{ij}, \varphi]$$

There will not be time to study the actions involving Ramond-Ramond fields, although these are extremely interesting too.

- On general grounds we can guess some properties of this action.

  For example, the gauge field should be a propagating field and its kinetic term must be gauge invariant and reparametrization invariant, therefore we expect a term
  $$\sim \int d^{10}x \; g^{ik} g^{jl} F_{ij} F_{kl}$$
  in $S_{\text{NS-NS}}$.

- The full action is not known. But there is a convenient approximation which reveals much of the dynamics, the approximation of slowly varying field strengths.

  In this approximation, we keep all powers of $F_{ij}$ itself, but drop its derivatives:
  $$l_s^3 \partial F, \; l_s^4 \partial\partial F, \; \ldots, \; l_s^{n+2} \partial^n F \ll 1$$
  where $l_s = \sqrt{2\pi\alpha'}$ is the string length.

- We will learn a lot about the open-string effective action in this approximation.

- We work in a closed-string background of constant $g, B, \varphi$. In particular, we take $B_{ij} \neq 0$ in all the 10 directions, and of maximal rank.

- The key result will be that open-string actions can be written in many different ways.

  One of these ways will look quite familiar, while the others will involve noncommutative multiplication of fields. Thus, the description of open-string actions involve a noncommutative algebra over spacetime, or noncommutative geometry.

  Whether we use the commutative or noncommutative formalism depends on the problem at hand.

  The existence of many different formalisms admits new insights into various issues in open-string theory.

## 2. Open String Effective Actions

- We start by deriving the action $S_{\text{NS-NS}}[A_i; g_{ij}, B_{ij}, \varphi]$ in conventional (commutative) variables.

(i) The Boundary Propagator

- Consider the Euclidean world-sheet action for an open string in a background gauge field $A_i$, and closed string backgrounds $g_{ij}, B_{ij}$. We will take the closed string fields to be constant for now.

The worldsheet is the upper half plane, with coordinates

$$\tau: \; -\infty < \tau < \infty; \qquad \sigma: \; 0 \leq \sigma < \infty$$

and we will also use $z = \tau + i\sigma$.

The action is:

$$S(A, g, B; X) = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau\;\, g_{ij}\, \partial_a X^i \partial_a X^j - \frac{i}{2} \int d\sigma\, d\tau\;\, B_{ij}\, \epsilon^{ab} \partial_a X^i \partial_b X^j$$
$$- \, i \int d\tau\;\, A_i(X)\, \partial_\tau X^i$$

We see that $g_{ij}, B_{ij}$ couple over the bulk of the worldsheet, while $A_i$ couples on the boundary at $\sigma = 0$.

- Under the transformation of the gauge field

$$\delta A_i = \partial_i \Lambda(X)$$

the action is invariant:

$$\delta \left( -i \int d\tau \; A_i(X)\, \partial_\tau X^i \right) = -i \int d\tau \; \partial_i \Lambda(X)\, \partial_\tau X^i = -i \int d\tau \; \partial_\tau \Lambda(X) = 0$$

This is the (linearized) spacetime gauge invariance for the $U(1)$ gauge field $A_i(X)$.

- There is also an invariance under the transformation:

$$\delta B_{ij} = \partial_i \Lambda_j - \partial_j \Lambda_i, \qquad \delta A_i = -\Lambda_i$$

This is easy to check. Note that both the open string field $A_i$ and the closed string field $B_{ij}$ transform together. This is important.

- The worldsheet equations of motion satisfied by $X$ are the Laplace equation:

$$\partial_z \partial_{\overline{z}} X^i = 0$$

together with the boundary condition:

$$\left[ g_{ij}\, \partial_\sigma X^j + 2\pi i \alpha' (B+F)_{ij}\, \partial_\tau X^j \right]_{\sigma=0} = 0$$

Notice that the dependence on $A_i$ and $B_{ij}$ arises only through the gauge-invariant combination $(B+F)_{ij}$.

- It is convenient to define:

$$\mathcal{F}_{ij} \;\equiv\; 2\pi\alpha'\,(B+F)_{ij}$$

in terms of which the boundary condition becomes:

$$\left[ g_{ij}\, \partial_\sigma X^j + i\, \mathcal{F}_{ij}\, \partial_\tau X^j \right]_{\sigma=0} = 0$$

or in terms of complex coordinates,

$$(g+\mathcal{F})_{ij}\, \partial_z X^j \Big|_{\sigma=0} = (g-\mathcal{F})_{ij}\, \partial_{\overline{z}} X^j \Big|_{\sigma=0}$$

- To study the quantized worldsheet theory, we compute the propagator:
$$\langle X^i(z,\bar z) X^j(z',\bar z')\rangle = K^{ij}(z,\bar z; z',\bar z')$$
This satisfies the differential equation
$$\partial_z \partial_{\bar z} K^{ij} = -2\pi\alpha' \, \delta^2(z-z')$$
along with the boundary condition:
$$(g+\mathcal{F})_{ij}\, \partial_z K^{jk}\Big|_{\sigma=0} = (g-\mathcal{F})_{ij}\, \partial_{\bar z} K^{jk}\Big|_{\sigma=0}$$
The general solution of the first equation is
$$K^{ij}(z,\bar z; z',\bar z') = -\alpha' \left( g^{ij} \ln|z-z'| + f^{ij}(z) + \bar f^{ij}(\bar z) \right)$$
where $f, \bar f$ are analytic functions of $z, \bar z$ respectively. These functions are then chosen to solve the boundary condition.

- It is an easy exercise to show that
$$(g+\mathcal{F})_{ij}\, f^{jk}(z) = \frac{1}{2}(g-\mathcal{F})_{ij}\, g^{jk} \ln(z-\bar z') + \text{const.}$$
$$(g-\mathcal{F})_{ij}\, \bar f^{jk}(\bar z) = \frac{1}{2}(g+\mathcal{F})_{ij}\, g^{jk} \ln(\bar z - z') + \text{const.}$$

Thus the propagator is:

$$K^{ij}(z,\overline{z};z',\overline{z}') = -\alpha'\Big[g^{ij}\ln|z-z'| + \frac{1}{2}\big\{(g+\mathcal{F})^{-1}(g-\mathcal{F})g^{-1}\big\}^{ij}\ln(z-\overline{z}') + \frac{1}{2}\big\{(g-\mathcal{F})^{-1}(g+\mathcal{F})g^{-1}\big\}^{ij}\ln(\overline{z}-z')\Big]$$

Let us evaluate this on the boundary of the string worldsheet, by taking $z=\tau+i\epsilon, z'=\tau'+i\epsilon$ with the limit $\epsilon\to 0$ taken at the end. Then, using:

$$\ln(\tau-\tau'+2i\epsilon) = \ln|\tau-\tau'| + i\pi\left(1-\theta(\tau-\tau')\right)$$
$$\ln(\tau-\tau'-2i\epsilon) = \ln|\tau-\tau'| + i\pi\,\theta(\tau-\tau')$$

we find:

$$K^{ij}(\tau;\tau') = -\alpha'\Big[2\left(\frac{1}{g+\mathcal{F}}\,g\,\frac{1}{g-\mathcal{F}}\right)^{ij}\ln|\tau-\tau'| + i\pi\left(\frac{1}{g+\mathcal{F}}\,\mathcal{F}\,\frac{1}{g-\mathcal{F}}\right)^{ij}\varepsilon(\tau-\tau')\Big]$$

where $\varepsilon(x) = +1\ (x>0),\ = -1\ (x<0)$.

- This formula has profound implications for open-string theory. It is useful to assign symbols for the tensors that appeared in the boundary propagator:

$$G^{ij}(\mathcal{F}) \equiv \left(\frac{1}{g+\mathcal{F}}\, g\, \frac{1}{g-\mathcal{F}}\right)^{ij}$$

$$\theta^{ij}(\mathcal{F}) \equiv -2\pi\alpha' \left(\frac{1}{g+\mathcal{F}}\, \mathcal{F}\, \frac{1}{g-\mathcal{F}}\right)^{ij}$$

  Then,

$$K^{ij}(\tau;\tau') = -2\alpha'\, G^{ij}(\mathcal{F}) \ln|\tau-\tau'| + \frac{i}{2}\,\theta^{ij}(\mathcal{F})\,\varepsilon(\tau-\tau')$$

- The tensor $G^{ij}(\mathcal{F})$ can be thought of as an "open-string metric", since it determines the log term in the boundary propagator.

  The tensor $\theta^{ij}$ gives rise to a sort of noncommutativity, as we will see shortly.

  We will now use this boundary propagator to determine the open-string effective action.

## (ii) Derivation of DBI Action

- To obtain the low-energy effective action for the gauge field, we need to expand the worldsheet theory about a background field $\overline{X}^i$, and compute the (divergent) one-loop counterterm:

$$-i \int d\tau \; \Gamma_i(A(X), \Lambda) \, \partial_\tau X^i$$

  where $\Lambda$ is an ultraviolet cutoff.

  Setting $\Gamma_i$ to zero gives a condition on the gauge field $A_i(X)$, equivalent to worldsheet conformal invariance, or vanishing of the $\beta$-function.

  That gives the spacetime equation of motion, from which the action can be reconstructed.

- Performing the background field expansion

$$X^i = \overline{X}^i + \xi^i$$

where $\overline{X}^i$ is an arbitrary classical solution of the worldsheet equations of motion, we find:

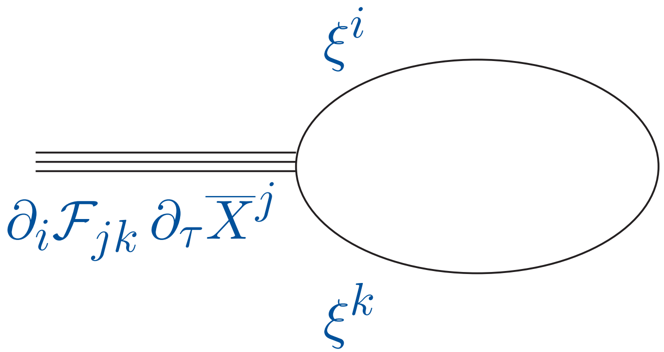
$$S(X) = S(\overline{X}) + \int \frac{\delta S}{\delta X^i}\Big|_{X=\overline{X}} \xi^i + \frac{1}{2}\int \frac{\delta^2 S}{\delta X^i \delta X^j}\Big|_{X=\overline{X}} \xi^i \xi^j + \cdots$$

The linear term in $\xi^i$ vanishes as $\overline{X}$ is a solution of the equation of motion.

The quadratic term is easily evaluated:

$$\frac{1}{2}\int \frac{\delta^2 S}{\delta X^i \delta X^j}\Big|_{X=\overline{X}} \xi^i \xi^j = \frac{1}{4\pi\alpha'}\Big[\int d\sigma d\tau\ g_{ij}\,\partial_a \xi^i \partial_a \xi^j$$
$$+\, i\int d\tau\ \left(\partial_i \mathcal{F}_{jk}\,\partial_\tau \overline{X}^j\,\xi^i \xi^k + \mathcal{F}_{ij}\,\xi^j \partial_\tau \xi^i\right)\Big]$$

This gives us a vertex for the following one-loop diagram:




- From this we find the correction to the worldsheet action:

$$\frac{i}{4\pi\alpha'} \int d\tau\ \partial_i \mathcal{F}_{jk}\, \partial_\tau \overline{X}^j\, K^{ik}(\tau, \tau')\Big|_{\tau=\tau'}$$

where we have ignored possible UV finite terms.

Recalling the formula for the boundary propagator, we have:

$$\lim_{\tau\to\tau'} K^{ij}(\tau;\tau') = -2\alpha' \, G^{ij}(\mathcal{F}) \, \ln\Lambda \;+\; (\text{finite})$$

and hence the equation of motion is:

$$\partial_i \mathcal{F}_{jk} \, G^{ik}(\mathcal{F}) \;\equiv\; \partial_i \mathcal{F}_{jk} \left( \frac{1}{g+\mathcal{F}} \, g \, \frac{1}{g-\mathcal{F}} \right)^{ik} = 0$$

- When we think of $G^{ij}(\mathcal{F})$ as the (inverse) open-string metric, then this looks just like the free Maxwell equations.

  But this does not mean the action is the Maxwell action, since $G^{ij}(\mathcal{F})$ depends on $\mathcal{F}$. We have to find an action that reproduces the full nonlinear equation above.

- It turns out that the desired open-string effective action is:

$$S_{\text{NS-NS}}[A_i; g_{ij}, B_{ij}] = \frac{1}{g_s} \int d^{10}x \, \sqrt{\det(g+\mathcal{F})}$$

It is a slightly lengthy exercise to show that:

$$\partial_i\left(\frac{\delta}{\delta(\partial_i A_j)}\sqrt{\mathsf{det}(g+\mathcal{F})}\right) \;=\; -\sqrt{\mathsf{det}(g+\mathcal{F})}\,G^{jk}(\mathcal{F})\,\partial_i\mathcal{F}_{kl}\,G^{li}(\mathcal{F})$$

Since the factor $\sqrt{\mathsf{det}(g+\mathcal{F})}$ is nonzero and the matrix $G^{jk}(\mathcal{F})$ is invertible, it follows that setting the above expression to zero is equivalent to:

$$\partial_i\mathcal{F}_{kl}\,G^{li}(\mathcal{F}) = 0$$

which is the desired equation of motion. At lowest order in $\mathcal{F}$ this is equivalent to the Maxwell equations:

$$\partial_i\mathcal{F}_{kl}\,g^{li} = 0$$

but in general, as we noted, it has nonlinear corrections.

- The action

$$S_{\text{NS-NS}}[A_i; g_{ij}, B_{ij}] = \frac{1}{g_s} \int d^{10}x \sqrt{\det(g + \mathcal{F})}$$
$$= \frac{1}{g_s} \int d^{10}x \sqrt{\det(g + 2\pi\alpha'(B + F))}$$

is called the Dirac-Born-Infeld (DBI) action.

Expanding this action to quadratic order in $F$, it is easily seen that it is proportional to the usual action of free Maxwell electrodynamics:

$$S_{\text{NS-NS}}[A_i; g_{ij}, B_{ij}] \sim \int F_{ij} F^{ij} + \cdots$$

This is consistent with the fact that the linearized equations of motion are just Maxwell's equations.

- For $\mathcal{F} = 0$, the DBI action reduces to:

$$S_{\text{NS-NS}}[A_i; g_{ij} = 0, B_{ij} = 0] = \frac{1}{g_s} \int d^{10}x \sqrt{\det g} = T\,V$$

where $V$ is the world-volume spanned by the D9-brane. This tells us that the coefficient of the action should be the tension of the brane:

$$T_p = \frac{1}{g_s\,(2\pi)^p\,(\alpha')^{\frac{p+1}{2}}}$$

We are ignoring the factors other than $\frac{1}{g_s}$, but they should always be understood to be present.

- Since $g_s = e^{\varphi}$ for constant dilaton $\varphi$, we have also fixed the dilaton dependence of the action.

## 3. Noncommutative Open String Actions

### (i) Rewriting the DBI Action

- We will now recast the DBI action in a different form. For this, let us first define

$$G^{ij} \equiv G^{ij}(F=0) = \left(\frac{1}{g+2\pi\alpha' B}\, g\, \frac{1}{g-2\pi\alpha' B}\right)^{ij}$$

$$\frac{\theta^{ij}}{2\pi\alpha'} \equiv \frac{\theta^{ij}(F=0)}{2\pi\alpha'} = -\left(\frac{1}{g+2\pi\alpha' B}\, 2\pi\alpha' B\, \frac{1}{g-2\pi\alpha' B}\right)^{ij}$$

We abbreviate $G^{ij}$ by $G^{-1}$. We also define the matrix $G_{ij}$, abbreviated $G$, to be the matrix inverse of $G^{ij}$.

Thus we have defined two new constant tensors $G^{-1}, \theta$ in terms of the original constant tensors $g, B$.

In particular,

$$\left(G^{-1} + \frac{\theta}{2\pi\alpha'}\right)^{ij} = \left(\frac{1}{g+2\pi\alpha' B}\right)^{ij}$$

- It is illuminating to rewrite the DBI Lagrangian in terms of the new tensors. This is achieved by writing:

$$\frac{1}{g_s}\sqrt{\det(g+2\pi\alpha'(B+F))} = \frac{1}{g_s}\sqrt{\det\left(\frac{1}{G^{-1}+\frac{\theta}{2\pi\alpha'}}+2\pi\alpha' F\right)}$$
$$= \frac{1}{g_s}\frac{1}{\sqrt{\det\left(1+\frac{G\theta}{2\pi\alpha'}\right)}}\sqrt{\det\left(G(1+\theta F)+2\pi\alpha' F\right)}$$
$$= \frac{1}{g_s}\frac{\sqrt{\det(1+\theta F)}}{\sqrt{\det\left(1+\frac{G\theta}{2\pi\alpha'}\right)}}\sqrt{\det\left(G+2\pi\alpha' F\frac{1}{1+\theta F}\right)}$$

  Defining

$$\hat{F} = F\frac{1}{1+\theta F}, \qquad G_s = g_s\sqrt{\det\left(1+\frac{G\theta}{2\pi\alpha'}\right)}$$

  we end up with the relation:

$$\frac{1}{g_s}\sqrt{\det(g+2\pi\alpha'(B+F))} = \frac{1}{G_s}\sqrt{\det(1+\theta F)}\sqrt{\det(G+2\pi\alpha'\hat{F})}$$

- The relation between $F$ and $\hat{F}$ can be easily inverted, leading to:

$$F = \hat{F}\frac{1}{1-\theta\hat{F}}$$

from which it also follows that:

$$1+\theta F = \frac{1}{1-\theta\hat{F}}$$

Hence we have:

$$\frac{1}{g_s}\sqrt{\det(g+2\pi\alpha'(B+F))} = \frac{1}{G_s}\frac{1}{\sqrt{\det(1-\theta\hat{F})}}\sqrt{\det(G+2\pi\alpha'\hat{F})}$$

In what follows, we must be careful to remember that the above equations were obtained in the strict DBI approximation of constant $F$.

- Apart from the factor $\sqrt{\det(1-\theta\hat{F})}$ in the denominator, which we will deal with later, the RHS looks like a DBI Lagrangian with a new string coupling $G_s$, metric $G_{ij}$ and gauge field strength $\hat{F}$, and no $B$-field.

  Let us therefore tentatively define the action:

$$\hat{S}_{\text{DBI}} = \frac{1}{G_s}\int\sqrt{\det(G+2\pi\alpha'\hat{F})}$$

## (ii) $\hat{F}$ as a Gauge Field Strength

- We will first try to understand this action for its own sake. Can $\hat{F}$ really be thought of as a gauge field strength? If so, what is its gauge potential?

  Let us start by expanding the relation through which we defined $\hat{F}$, to lowest order in $\theta$:

$$\hat{F}_{ij} = F_{ij} - F_{ik}\,\theta^{kl}F_{lj} + \mathcal{O}(\theta^2)$$

Inserting the definition of $F_{ij}$, we get:

$$\hat{F}_{ij} = \partial_i A_j - \partial_j A_i \\ + \theta^{kl}(\partial_i A_k \, \partial_j A_l - \partial_i A_k \, \partial_l A_j - \partial_k A_i \, \partial_j A_l + \partial_k A_i \, \partial_l A_j) \\ + \mathcal{O}(\theta^2)$$

We can make a nonlinear redefinition of $A_i$ to this order, which absorbs three of the four terms linear in $\theta$:

$$\hat{A}_i = A_i - \theta^{kl}(A_k \, \partial_l A_i + \frac{1}{2} A_k \, \partial_i A_l) + \mathcal{O}(\theta^2)$$

Indeed, it is easy to show that

$$\partial_i \hat{A}_j - \partial_j \hat{A}_i = \partial_i A_j - \partial_j A_i \\ + \theta^{kl}(\partial_i A_k \, \partial_j A_l - \partial_i A_k \, \partial_l A_j - \partial_k A_i \, \partial_j A_l - A_k \, \partial_l F_{ij})$$

Note that we should drop the last term here, proportional to $\partial_l F_{ij}$, because we are working in the approximation of constant $F$.

With the above definition, we find that

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \theta^{kl} \partial_k \hat{A}_i \, \partial_l \hat{A}_j$$

(in the last term we have replaced $A$ by $\hat{A}$, since we are working to linear order in $\theta$.)

It is clear that there is no further redefinition of $\hat{A}$ that will absorb the last term. However, we note that this term is:

$$\theta^{kl} \partial_k \hat{A}_i \, \partial_l \hat{A}_j = \{\hat{A}_i, \hat{A}_j\}$$

where $\{,\}$ is the Poisson bracket with Poisson structure $\theta$.

- Thus, to linear order in $\theta$, we have found that:

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i + \{\hat{A}_i, \hat{A}_j\}$$

This looks like a non-Abelian gauge field strength, except that there is a Poisson bracket instead of a commutator.

In mechanics, we know that the Poisson bracket of classical theory lifts to a commutator in quantum theory. Something similar happens here.

- A theorem of Kontsevich (stated informally) says that a Poisson bracket of this kind can be lifted in a unique way to a commutator under noncommutative multiplication:

$$\{f,g\} \Rightarrow -i(f*g-g*f)$$

where

$$f(x)*g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)\,g(y)\Big|_{x=y}$$

This is called the Moyal-Weyl product. It is a noncommutative but associative product.

By "lifting" we mean going from linear order in $\theta$ to all orders. Because there is a unique way to perform this lift, there must be a unique map from $A$ to $\hat{A}$ which solves the equation (for constant $F$):

$$\hat{F} = F\frac{1}{1+\theta F}$$

to all orders in $\theta$.

- We conclude that the new field strength $\hat{F}_{ij}$ is a noncommutative field strength related to its gauge potential $\hat{A}_i$ by:

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i[\hat{A}_i, \hat{A}_j]_*$$

The map

$$\hat{F}_{ij} = F_{ik} \left( \frac{1}{1 + \theta F} \right)^k{}_j$$

$$\hat{A}_i = A_i - \theta^{kl} (A_k \, \partial_l A_i + \frac{1}{2} A_k \, \partial_i A_l) + \mathcal{O}(\theta^2)$$

is known as the Seiberg-Witten map.

As already indicated, we have only considered this map in the DBI approximation of constant field strength $F_{ij}$.

Later we will explain how to go beyond this approximation.

## (iii) The Prefactor

- To complete the story of the DBI action we must understand the prefactor that we ignored:

$$\sqrt{\det(1 - \theta\hat{F})}$$

  Unless we find an explanation for this, we cannot claim that commutative and non-commutative DBI actions are physically equivalent.

- Actually, for constant $F$, all the actions we have been writing are infinite. So we cannot really compare commutative and noncommutative actions.

  If we want fields to fall off fast enough at infinity, we must allow them to vary. The explanation for this factor will emerge only when we consider varying fields, or equivalently, modes of nonzero momentum.

  That in turn will come about when we allow the relevant closed-string background, in this case the dilaton, to vary.

  To do this, we need to understand carefully the nature of gauge invariance in noncommutative gauge theory, to which we now turn our attention.

## 4. Properties of Noncommutative Gauge Theory

- In this section we focus our attention on gauge theories where multiplication is defined through the $*$-product:

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}} f(x)\, g(y)\Big|_{x=y}$$
$$= f(x)\, g(x) + \frac{i}{2}\theta^{ij}\,\partial_i f(x)\,\partial_j g(x) + \cdots$$

Some formal properties of this product are:

$$\begin{aligned} (i) &\quad f * g \neq g * f \\ (ii) &\quad f * (g * h) = (f * g) * h \\ (iii) &\quad \int dx\ f * g = \int dx\ g * f = \int dx\ fg \\ (iv) &\quad \int dx\ [f, g]_* = 0 \end{aligned}$$

The last two properties clearly require the integrand to fall off sufficiently fast at infinity.

- The simplest such theory is noncommutative electrodynamics:

$$S = -\frac{1}{4}\int dx\ \hat{F}_{ij} * \hat{F}^{ij} = -\frac{1}{4}\int dx\ \hat{F}_{ij}\, \hat{F}^{ij}$$

  where

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i[\hat{A}_i, \hat{A}_j]_*$$

  Indeed, this action arises from the noncommutative DBI action

$$\sqrt{\det(G + 2\pi\alpha' \hat{F})}$$

  if we expand it to quadratic order in $\hat{F}$.

- Noncommutative electrodynamics is different from noncommutative Yang-Mills theory. In the latter case, $\hat{F}$ would be $N \times N$ matrices, besides having the $*$-product structure. We will have little to say about this situation in these lectures.

- What is the gauge transformation for noncommutative electrodynamics? We easily guess that it must be:

$$\delta \hat{A}_i = \partial_i \Lambda + i[\Lambda, A_i]_*$$

Using manipulations familiar from Yang-Mills theory, we find that

$$\delta \hat{F}_{ij} = i[\Lambda, \hat{F}_{ij}]_*$$

Then, the gauge variation of the action is:

$$\delta S = -\frac{1}{4} \int dx \; [\Lambda, \hat{F}_{ij} * \hat{F}^{ij}]_* = 0$$

Note that the Lagrangian is not gauge invariant, only the action is invariant. Gauge invariance comes about after performing the integral.

This is reminiscent of the fact that in non-Abelian gauge theory, gauge invariance is achieved only after taking the trace.

Apparently the integral over non-commutative fields is like a trace.

- We can gain some insight into this by noting that with the $*$-product, the coordinates of spacetime satisfy:

$$[x^i, x^j]_* = i\,\theta^{ij}$$

Thus, noncommutative field theory can be thought of as field theory on a noncommutative spacetime.

In this formalism, we would promote the coordinates to operators $\hat{\mathsf{x}}^i$ satisfying:

$$[\hat{\mathsf{x}}^i, \hat{\mathsf{x}}^j] = i\,\theta^{ij}$$

There would no longer be a Moyal product, instead noncommutativity would be due to the coordinates being noncommuting operators. Here we will stick to the Moyal product notation, with the coordinates being ordinary numbers.

- Let us now show that the generators of noncommutative gauge transformations are just translations. First, observe that:

$$e^{ia\cdot x} * x^j = (x^i + \theta^{ji} a_i) * e^{ia\cdot x}$$

On general functions we have:

$$e^{ia\cdot x} * f(x) = f(x + \theta a) * e^{ia\cdot x}$$

Now we can recast the noncommutative gauge transformation

$$\delta \hat{F}_{ij}(x) = i\Big[\Lambda, \hat{F}_{ij}\Big]_*$$

as follows. Define the Fourier transform of the gauge parameter by:

$$\Lambda(x) = \int dk\ e^{ik\cdot x}\ \widetilde{\Lambda}(k)$$

Then,

$$\begin{aligned}[\Lambda(x), \hat{F}_{ij}(x)] &= \int dk\ \widetilde{\Lambda}(k)\ \Big[e^{ik\cdot x}, \hat{F}_{ij}(x)\Big] \\ &= \int dk\ \widetilde{\Lambda}(k)\ \Big(\hat{F}_{ij}(x + \theta k) - \hat{F}_{ij}(x)\Big)\ e^{ik\cdot x}\end{aligned}$$

- This fact gives us an important insight into why the Lagrangian of noncommutative gauge theory is not gauge invariant.

  It will also provide the solution to this and other difficulties, including ultimately the role of the prefactor in the DBI action.

## 5. Open Wilson Lines

- Let us exhibit in a slightly different language the fact that only the zero-frequency mode of the noncommutative Lagrangian is gauge invariant, while the other modes are not.

- Define the Fourier modes of a gauge theory Lagrangian by

$$S(k) = \int dx\, \hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{ik.x}$$

Let the finite gauge transformations be given by:

$$U(x) = \Big(e^{i\Lambda(x)}\Big)_*$$

where the $*$ subscript means that the exponential is defined as a power series with $*$-products.

If the Lagrangian is appropriately constructed from $\hat{F}_{ij}$, it will transform as:

$$\hat{\mathcal{L}}\big(\hat{A}(x)\big) \to U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * U^{-1}(x)$$

It follows that

$$
\begin{aligned}
S(k) \to \int dx\, U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * U^{-1}(x) * e^{ik.x} \\
= \int dx\, U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{ik.x} * U^{-1}(x - \theta k) \\
= \int dx\, U^{-1}(x - \theta k) * U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{ik.x}
\end{aligned}
$$

Clearly the effect of the gauge transformation cancels only at $k = 0$.

This is not really a big surprise. Because noncommutative gauge transformations are spacetime translations, there are no local gauge-invariant observables in noncommutative gauge theory.

But all is not lost. Even though we cannot make gauge-invariant functions of $x$, we can make gauge-invariant functions of the Fourier momenta $k$.

- The inspiration for this comes from a physical notion. In non-Abelian gauge theories, the Wilson line is a non-local operator

$$W(C) = P\,\mathrm{exp}\left(-i\int_C A_i(x)\,dx^i\right)$$

Here the gauge field $A_i(x)$ is an $N \times N$ matrix, and therefore $W(C)$ is also a matrix.

$C$ denotes a contour in spacetime, which can be either open or closed.

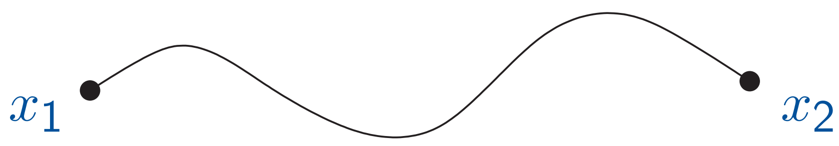
$P$ denotes path-ordering, which means that we multiply the exponentials over little bits of the path. If the path is broken into infinitesimal pieces

$$C = C_1 \cup C_2 \cup \cdots \cup C_n \qquad n \to \infty$$

then:

$$P\,e^{-i\int_C A_i(x)\,dx^i} \equiv \lim_{n\to\infty} e^{-i\int_{C_1} A_i(x)\,dx^i} e^{-i\int_{C_2} A_i(x)\,dx^i} \ldots e^{-i\int_{C_n} A_i(x)\,dx^i}$$

- Let us first choose $C$ to be an open contour, from $x_1$ to $x_2$:




Then, under a (finite) local gauge transformation by $U(x)$, under which

$$A_i(x) \;\rightarrow\; i\,U(x)\,\partial_i U^{-1}(x) + U(x)\,A_i(x)\,U^{-1}(x)$$

we have

$$W(C) \rightarrow U(x_1)\,W(C)\,U^{-1}(x_2)$$

We now briefly recall why $W(C)$ transforms this way.

- Take an infinitesimal contour, $C \sim \Delta x^i$. Then

$$
\begin{aligned}
W(\Delta x) &= \exp\left(-i\, A_i(x)\, \Delta x^i\right) \\
&\sim 1 - i\, A_i(x)\, \Delta x^i \\
&\to 1 - i\left(i\, U(x)\, \partial_i U^{-1}(x) + U(x)\, A_i(x)\, U^{-1}(x)\right)\Delta x^i \\
&\sim U(x)\left(1 - i\, A_i(x)\, \Delta x^i\right) U^{-1}(x + \Delta x) \\
&\sim U(x)\, W(\Delta x)\, U^{-1}(x + \Delta x)
\end{aligned}
$$

Multiplying these factors over every infinitesimal piece $C_1, C_2, \cdots, C_n$ of a finite contour gives the desired result. It is clear from this why we need the path ordering.

The above result also tells us that $\mathrm{tr}\, W(C)$ is gauge invariant only for closed contours, $x_1 = x_2$.

- The analogous story goes through easily for noncommutative gauge theory.

  This time we use the gauge field $\hat{A}_i(x)$, which transforms under noncommutative gauge transformations as:

$$\hat{A}_i(x) \;\rightarrow\; i\, U(x) * \partial_i U^{-1}(x) + U(x) * \hat{A}(x) * U^{-1}(x)$$

  Under these transformations, the Wilson line

$$W(C) = P_* \,\mathsf{exp}\left(-i\int_C \hat{A}_i(x)\, dx^i\right)$$

  transforms to:

$$W(C) \;\rightarrow\; U(x_1) * W(C) * U^{-1}(x_2)$$

- In the next section we make use of this result by starting with an open Wilson line, and using its lack of gauge invariance to compensate for the gauge non-invariance of the noncommutative action.

## 6. Gauge Invariant Noncommutative Actions

- Let us put together two ingredients.

  On one hand, we have the momentum-$k$ mode of a noncommutative gauge theory Lagrangian, which fails to be gauge invariant:

$$\int dx\, \hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{ik.x} \;\rightarrow\; \int dx\, U^{-1}(x-\theta k) * U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{ik.x}$$

  On the other, we have an open Wilson line

$$W(C) = W(x_1, x_2) = P_* \,\mathsf{exp}\left(i\int_{x_1}^{x_2} \hat{A}_i(x)\, dx^i\right)$$

  which also fails to be gauge invariant:

$$W(C) \;\rightarrow\; U(x_1) * W(C) * U^{-1}(x_2)$$

  Suppose we combine the two.

A natural object that one can form from both of them is the product:

$$\int dx\, \hat{\mathcal{L}}\big(\hat{A}(x)\big) * W(C) * e^{ik.x} = \int dx\, \hat{\mathcal{L}}\big(\hat{A}(x)\big) * P_* \, e^{-i\int_{x_1}^{x_2} \hat{A}_i \, dx^i} * e^{ik.x}$$

Under a noncommutative gauge transformation, this goes to:

$$\int dx\, U(x) * \hat{\mathcal{L}}\big(\hat{A}(x)\big) * U^{-1}(x) * U(x_1) * P_* \, e^{-i\int_{x_1}^{x_2} \hat{A}_i \, dx^i} * U^{-1}(x_2) * e^{ik.x}$$

If we choose the contour to start at $x_1^i = x^i$ and end at $x_2^i = x^i + \theta^{ij} k_j$ then we see that the above is gauge invariant for every $k$.

- In principle this still allows for any shape of the contour. But we will choose the simplest one, a straight line:


and justify it later.

- One final step is required before we can write the gauge invariant noncommutative action for open strings. In the above, the Lagrangian is evaluated at the starting point of the Wilson line. Pictorially:


which corresponds to the term

$$\hat{\mathcal{L}}\Big(\hat{A}(x)\Big) * P_* \, e^{-i \int_x^{x+\theta k} \hat{A}_i \, dx^i}$$

But we could equally well insert $\hat{\mathcal{L}}$ at any other point $x'^i$ along the Wilson line, as long as we keep it inside the path ordering sign:


This would correspond to:

$$P_* \left\{ e^{-\int_x^{x+\theta k} \hat{A}_i \, dx^i} * \hat{\mathcal{L}}\Big(\hat{A}(x')\Big) \right\}$$

and would also give a gauge invariant action $S(k)$.

- The most democratic option is to smear every operator in $\hat{\mathcal{L}}$ along the contour of the open Wilson line.

  Suppose for example that

$$\hat{\mathcal{L}}\big(\hat{A}(x)\big) = -\frac{1}{4}\hat{F}_{ij}(x) * \hat{F}^{ij}(x)$$

  then the smeared version is:

$$-\frac{1}{4}\int dx \int_0^1 d\tau_1 \int_0^1 d\tau_2 \, P_* \Big\{\hat{F}_{ij}(x^i + \theta^{ij}k_j\tau_1) * \hat{F}^{ij}(x^i + \theta^{ij}k_j\tau_2) * \exp(-i\int_x^{x+\theta k} \hat{A}_i \, dx^i\Big\}$$

  As $\tau_1$ and $\tau_2$ vary from $0$ to $1$, each operator in the Lagrangian gets smeared over the location of the Wilson line. The result is gauge invariant as before, since everything is inside path ordering.

  Introducing the notation $L_*$ to denote the combined operation of smearing and path ordering, we can write the above as:

$$-\frac{1}{4}\int dx \; L_* \Big\{\hat{F}_{ij}(x) * \hat{F}^{ij}(x) * e^{-i\int_x^{x+\theta k} \hat{A}_i \, dx^i}\Big\} * e^{ik.x}$$

- This gives us our final prescription to write a gauge invariant “action” (as a function of the momentum $k$) for every local gauge covariant Lagrangian $\hat{\mathcal{L}}\big(\hat{A}(x)\big)$ in noncommutative gauge theory:

$$\int dx\ L_*\left\{\hat{\mathcal{L}}\big(\hat{A}(x)\big) * e^{-i\int_x^{x+\theta k}\hat{A}_i\,dx^i}\right\} * e^{ik.x}$$

Applying this to the Dirac-Born-Infeld Lagrangian:

$$\hat{\mathcal{L}}_{\mathrm{DBI}}\big(\hat{A}(x)\big) = \frac{1}{G_s}\sqrt{\det(G + 2\pi\alpha'\hat{F})}$$

we get the noncommutative DBI action:

$$\hat{S}_{\mathrm{DBI}}(k) = \frac{1}{G_s}\int dx\ L_*\left\{\sqrt{\det(G + 2\pi\alpha'\hat{F})} * e^{-i\int_x^{x+\theta k}\hat{A}_i\,dx^i}\right\} * e^{ik.x}$$

- Let us recall our earlier discussion of the DBI action. The Seiberg-Witten map between commutative and noncommutative gauge fields, in the strict DBI approximation, was written:

$$F = \hat{F}\frac{1}{1-\theta\hat{F}}$$

We had also obtained the relation:

$$\frac{1}{G_s}\sqrt{\det(G+2\pi\alpha'\hat{F})} = \frac{1}{g_s}\sqrt{\det(1-\theta\hat{F})}\sqrt{\det\left(g+2\pi\alpha'\left(B+\hat{F}\frac{1}{1-\theta\hat{F}}\right)\right)}$$

The second term on the RHS is the commutative DBI action expressed in terms of $\hat{F}$, and we had promised an explanation of the prefactor $\sqrt{\det(1-\theta\hat{F})}$, which is now about to emerge.

- This relation gives us an alternative form of the proposed noncommutative DBI action:

$$\hat{S}_{\mathrm{DBI}}(k) =$$
$$\frac{1}{g_s}\int dx\ L_*\left\{\sqrt{\det(1-\theta\hat{F})} * \sqrt{\det\left(g+2\pi\alpha'\left(B+\hat{F}\frac{1}{1-\theta\hat{F}}\right)\right)} * e^{-i\int_x^{x+\theta k}\hat{A}_i\,dx^i}\right\} * e^{ik.x}$$

As a recipe, this is clear: start with the commutative action, replace $F$ in terms of $\hat{F}$, and insert two extra factors, $\sqrt{\det(1-\theta\hat{F})}$ and the open Wilson line. Finally, perform the $L_*$ operation over everything.

We can now announce our principal result:

- Claim: The above action is equal to the commutative DBI action:

$$S_{\mathrm{DBI}}(k) = \frac{1}{g_s}\int dx\ \sqrt{\det(g+2\pi\alpha'(B+F))}\ e^{ik.x}$$

under an appropriate Seiberg-Witten map, in the approximation of slowly varying fields, and

- In particular, this claim justifies our various assumptions, such as the fact that we took a straight open Wilson line, and that we took the democratic option of smearing all operators over the straight contour.

- Let us now turn to the extra factor

$$\sqrt{\mathrm{det}(1-\theta\hat{F})}$$

that we originally found when relating commutative and noncommutative actions.

Suppose the field strengths are all strictly constant. Then the expression on the previous page can be written:

$$\hat{S}_{\mathrm{DBI}}(k) = \frac{1}{g_s}\sqrt{\mathrm{det}\left(g + 2\pi\alpha'\left(B + \hat{F}\frac{1}{1-\theta\hat{F}}\right)\right)} \times$$
$$\int dx\ L_* \left\{\sqrt{\mathrm{det}(1-\theta\hat{F})} * e^{-i\int_x^{x+\theta k}\hat{A}_i\,dx^i}\right\} * e^{ik.x}$$

Amazingly, the second line is equal to $\delta(k)$! In other words the open Wilson line cancels out the effect of the prefactor, leaving behind the first line which is the commutative DBI action.

- To be more precise, it is an exact result in noncommutative gauge theory that

$$\int dx\ L_* \left\{ \sqrt{\det(1-\theta\hat{F})} * e^{-i\int_x^{x+\theta k} \hat{A}_i\, dx^i} \right\} * e^{ik.x} = \delta(k)$$

This result, the "topological identity", actually holds for all $F$, and not just in the DBI approximation.

It is a purely mathematical property of noncommutative gauge fields, and not of any particular action.

It is quite a subtle identity. Naively, the open Wilson line itself reduces to 1 when we take $k \to 0$. But the identity says this is not quite true. We get 1 only if we first multiply by the prefactor $\sqrt{\det(1-\theta\hat{F})}$ and then take $k \to 0$.

Unfortunately, there is no elementary proof of this identity. The proofs in the literature rely on the Chern-Simons action, or on the relation of noncommutativity to matrix theory, neither of which we have explored here.

So, for these lectures it is stated without proof, though we may return to it later.

- To summarize, we have found a gauge invariant noncommutative action for every momentum $k$, which is equal to the commutative action in the approximation of slowly varying (not necessarily constant) $F$.

  This action should physically be thought of as the coupling of open-string fields to a varying dilaton of momentum $k$.

  In the final section, we will briefly touch upon a few important directions raised by the study of noncommutativity.

## 7. Summary of Further Directions

### (i) Freedom in the Description

- The fundamental relation that we used to define $G_{ij}$ and $\theta^{ij}$ was:

$$\left(\frac{1}{G}\right)^{ij} + \frac{\theta^{ij}}{2\pi\alpha'} = \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}$$

This expression arose naturally, but it is not unique. Seiberg and Witten showed that one can have a family of noncommutative descriptions starting with the more general relation:

$$\left(\frac{1}{G + 2\pi\alpha'\Phi}\right)^{ij} + \frac{\theta^{ij}}{2\pi\alpha'} = \left(\frac{1}{g + 2\pi\alpha' B}\right)^{ij}$$

where $\Phi_{ij}$ is an antisymmetric tensor called the "description parameter".

From this point of view, we have been working in the $\Phi = 0$ description.

- In fact, there is a noncommutative DBI action for every $\Phi$. In particular, consider the choice:

$$\Phi_{ij} = B_{ij}$$

Inserting this into the defining relation, we see that in this case,

$$\theta^{ij} = 0, \quad G_{ij} = g_{ij}$$

In other words, we have obtained the commutative description.

- Another interesting choice is

$$\Phi_{ij} = -B_{ij}$$

for which we find

$$\frac{\theta^{ij}}{2\pi\alpha'} = \left(\frac{1}{B}\right)^{ij}$$

- From this we learn that the description parameter continuously interpolates between the commutative and various noncommutative descriptions.

We can now imagine varying $\theta$ with fixed physical backgrounds $g, B$, simply by varying $\Phi$. This makes manifest that noncommutativity is an option.

However, $\theta$ is always 0 when $B = 0$.

- From the relations

$$\hat{F} = F \frac{1}{1 + \theta F}, \qquad F = \hat{F} \frac{1}{1 - \theta \hat{F}}$$

we see that $\theta$ should always be chosen to avoid having

$$F = -\theta^{-1}$$

where the noncommutative description breaks down. Likewise, for

$$\hat{F} = \theta^{-1}$$

it is the commutative description that breaks down. For generic field strengths, both descriptions are simultaneously valid.

(ii) Derivative Corrections

- We have worked in the DBI approximation. Clearly it is interesting to go beyond that.

  In fact, noncommutativity gives us information about the derivative corrections to the DBI action.

  This works as follows. Suppose the full open-string action in the commutative description is

$$S_{\text{DBI}} + \Delta S_{\text{DBI}}$$

  where the second term contains all the derivative corrections.

  Similarly, in the noncommutative description we have:

$$\hat{S}_{\text{DBI}} + \Delta \hat{S}_{\text{DBI}}$$

If we believe the two descriptions are equivalent at a fundamental level, we expect the exact equality:

$$S_{\mathrm{DBI}} + \Delta S_{\mathrm{DBI}} = \hat{S}_{\mathrm{DBI}} + \Delta \hat{S}_{\mathrm{DBI}}$$

However, because the $*$-product involves derivatives, there is not an exact equality of $S_{\mathrm{DBI}}$ and $\hat{S}_{\mathrm{DBI}}$. In fact, we have argued that:

$$S_{\mathrm{DBI}} = \hat{S}_{\mathrm{DBI}} + \mathcal{O}(\partial F)$$

Remarkably, there is a limit in which derivative corrections are completely suppressed only on the noncommutative side :

$$\alpha' \sim \sqrt{\epsilon} \to 0, \qquad g_{ij} \sim \epsilon, \qquad B_{ij} \text{ fixed}$$

This is called the Seiberg-Witten limit. It is easy to check that in this limit, $G_{ij}$ remains finite.

- As the derivative expansion is an expansion in powers of $\alpha'$, one would expect all corrections to vanish as $\alpha' \to 0$.

  This expectation holds in the noncommutative case because $G^{ij}$ is used to contract tensors, and it remains fixed. But on the commutative side, $g^{ij}$ is used to contract tensors, and it becomes singular.

  As a result, infinitely many derivative corrections survive the Seiberg-Witten limit. We then have:

$$[\Delta S_{\mathsf{DBI}}]_{\mathsf{SW\ limit}} = [\hat{S}_{\mathsf{DBI}}]_{\mathsf{SW\ limit}} - [S_{\mathsf{DBI}}]_{\mathsf{SW\ limit}}$$

  Indeed, all the corrections surviving in the LHS are encapsulated in noncommutative $*$-products!

- This suggests that noncommutativity could give us a complete understanding of open-string actions beyond the DBI approximation – an important program that needs to be completed.

## (iii) Couplings to RR Fields

- Let us briefly consider the couplings of open-string modes to Ramond-Ramond fields. At the commutative level, these are expressed in the action:

$$S_{\text{R-R}}[A_i; C^{(2p)}_{i_1 i_2 \dots i_{2p}}] = \frac{1}{g_s} \int \sum_{r=0}^{5} C^{(2r)} \wedge e^{\mathcal{F}}$$

where wedge products are intended. This action is topological in that it does not depend explicitly on a metric. It is also called the Chern-Simons action, $S_{\text{CS}}$.

The notation above can be made more explicit by writing:

$$S_{\text{CS}} = \frac{1}{g_s} \int \left( C^{(10)} + C^{(8)} \wedge \mathcal{F} + \frac{1}{2} C^{(6)} \wedge \mathcal{F} \wedge \mathcal{F} + \cdots \right)$$

where of course, $\mathcal{F} \equiv 2\pi\alpha'(B + F)$.

By analogy with the DBI case, the noncommutative Chern-Simons action can be written:

$$\hat{S}_{CS}(k) = \frac{1}{g_s}\sum_{r=0}^{5} C^{(2r)} \wedge \int L_* \left\{ \sqrt{\det(1-\theta\hat{F})} * e^{\left(B+\hat{F}\frac{1}{1-\theta\hat{F}}\right)} * e^{-i\int_x^{x+\theta k} \hat{A}_i dx^i} \right\} * e^{ik.x}$$

In this case we can actually extract new information from the proposed equivalence of commutative and noncommutative actions.

This is because we know from independent calculations that the 10-form and 8-form RR couplings do not receive derivative corrections.

This leads to two identities:

$$(i)\ \int dx\ L_* \left\{ \sqrt{\det(1-\theta\hat{F})} * e^{-i\int_x^{x+\theta k} \hat{A}_i\, dx^i} \right\} * e^{ik.x} = \delta(k)$$

which is our old friend the topological identity, and

$$(ii)\ \int dx\, L_* \left\{ \sqrt{\det(1-\theta\hat{F})} * \left(\hat{F}\frac{1}{1-\theta\hat{F}}\right)_{ij} * e^{-i\int_x^{x+\theta k}\hat{A}_i\,dx^i} \right\} * e^{ik.x} = F_{ij}(k)$$

This is an exact expression for the Seiberg-Witten map beyond the DBI approximation!

Much more can be said, but we will have to leave this topic here.

## (iv) Noncommutative Solitons

- Besides the stable branes in type II superstring theory, there are also unstable branes that can decay into lower-dimensional branes or into the vacuum.

  This decay is described by a tachyon on the unstable brane going into its vacuum or into a solitonic configuration.

  In general, tachyonic solitons are hard to study because we do not know enough about the detailed form of the tachyon potential. However, if we turn on a $B$-field along the unstable brane and switch to a noncommutative description, life becomes much easier.

  The tachyon is now a noncommutative field, and its solitons have a universal description that is largely independent of the shape of its potential.

  It has been possible to give a rather explicit description of the decay of unstable branes using this idea. This is one more concrete application of noncommutativity in string theory.

## (v) Nonabelian Noncommutativity

- In these lectures, we only discussed the noncommutative versions of Abelian theories.

  We should try to generalize these discussions to the non-Abelian case. In that case, despite some progress, the explicit form of the DBI/CS actions, the topological identity and the Seiberg-Witten map are not yet known.

  This is an important open problem.

# THE END